On global anomalies of heterotic string theories

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Introduction

String theorists often say

String theory is miraculously free of inconsistency

and/or

String theory is the only consistent theory of quantum gravity

Are you really sure?

Really?

Really?

Today I would like to discuss
global anomalies of **perturbative heterotic string theories**.

It sounds esoteric.

But no, actually it isn't.

Recall Witten's $SU(2)$ anomaly.

In 4d, there is a nontrivial large gauge transformation associated to

$$\pi_4(SU(2)) = \mathbb{Z}_2.$$

If you have N_2 doublet Weyl fermions, this produces a phase

$$(-1)^{N_2}.$$

More generally, with N_k Weyl fermions in the k-dim'l irrep of $SU(2)$, the anomaly is

$$(-1)^{N_2+N_6+N_{10}+\cdots}.$$

Let us now ask:

Is Witten's $SU(2)$ anomaly absent in 4d heterotic compactifications?

The worldsheet theory consists of

	c_L	c_R
(super)ghosts	-26	-15
$X^{0,1,2,3}, \bar{\psi}^{0,1,2,3}$	4	6
internal CFT	22	9

My convention is that the right-movers are supersymmetric.

Massless fermions come from states of the internal CFT where:

- The right-movers are put into R-sector,
- Massless fermions are those with $(L_0, \bar{L}_0) = (1, 0)$,
- $(-1)^{F_R}$ translates to the spacetime chirality via GSO.

So the question is:

Take an $SU(2)$-symmetric $\mathcal{N}{=}(0,1)$ SCFT with $(c_L, c_R) = (22, 9)$.

Let N_k be the number of states in the k-dim'l irrep of $SU(2)$ with $(L_0, \bar{L}_0) = (1, 0)$ in the R-sector.

Is $(-1)^{N_2+N_6+N_{10}+\cdots} = +1$?

This is a nontrivial question which I don't know how to address using our standard toolkit.

I learned this question from [Enoki-Sato-Watari 2005.01069], written by my colleagues in IPMU.

Or more precisely, during a chat during one of the last tea time before the pandemic, when they were working on that paper.

They studied this question assuming that
there are 4d $\mathcal{N}=2$ spacetime SUSY,
and showed that the anomaly vanishes in many cases.

They couldn't find any counterexample,
but they didn't find a proof either.

My point today is that the mathematical theory of **topological modular forms** can address this question, assuming the validity of the **Segal-Stolz-Teichner conjecture**.

The list of hep-th papers on **tmf** is not very long.

The exhaustive list is

Gaiotto, Johnson-Freyd	1811.00589
Gukov, Pei, Putrov, Vafa	1811.07884
Gaiotto, Johnson-Freyd, Witten	1902.10249
Gaiotto, Johnson-Freyd	1904.05788
Johnson-Freyd	2006.02922
YT	2103.12211

It's a young field and new comers are welcomed...

(Texts in purple is hyperlinked if you download the slides.)

Take an $SU(2)$*-symmetric* $\mathcal{N}=(0,1)$ *SCFT*
with $(c_L, c_R) = (22, 9)$.

Let N_k *be the number of states in the* k*-dim'l irrep of* $SU(2)$
with $(L_0, \bar{L}_0) = (1, 0)$ *in the R-sector.*

Is $(-1)^{N_2+N_6+N_{10}+\cdots} = +1$ *?*

Although I am going to concentrate on the approach using tmf,
I believe this question should also have a more conventional answer.

Please come up with one.

Green-Schwarz mechanism

I would like to start with some refresher.

Superstring theory as we know it started with Green-Schwarz (1984).

In 10d $\mathcal{N}=1$ supergravity theory,
the anomaly polynomial of the fermions factorizes:

$$I_{12} = I_4 I_8, \qquad I_4 = \mathrm{tr}\, R^2 - \mathrm{tr}\, F^2$$

if

$$G = SO(32) \quad \text{or} \quad E_8 \times E_8$$

(I put all nontrivial coefficients into the normalization of tr.)

We posit that the gauge-invariant field strength H of the B field is

$$H = dB + CS(\omega) - CS(A)$$

where ω is the spin connection, A is the gauge field,
CS is the Chern-Simons form, such that

$$dH = I_4 = \mathrm{tr}\, R^2 - \mathrm{tr}\, F^2.$$

This makes B transform nontrivially under both

- the general coordinate transformation
- and the G gauge transformation.

We also posit the interaction

$$-\int_{M_{10}} B \wedge I_8.$$

This produces the anomaly

$$-(dH)I_8 = -I_4 I_8,$$

cancelling the anomaly from the fermions, which was $I_{12} = I_4 I_8$.

The explicit form of I_8 is complicated ...

Let's look up Polchinski ...

The chiral fields of $N = 1$ supergravity with gauge group g are the gravitino **56**, a neutral fermion **8**′, and an **8** gaugino in the adjoint representation, for total anomaly

$$\begin{aligned}
\hat{I}_{\mathrm{I}} &= \hat{I}_{\mathbf{56}}(R_2) - \hat{I}_{\mathbf{8}}(R_2) + \hat{I}_{\mathbf{8}}(F_2, R_2) \\
&= \frac{1}{1440}\left\{-\mathrm{Tr_a}(F_2^6) + \frac{1}{48}\mathrm{Tr_a}(F_2^2)\mathrm{Tr_a}(F_2^4) - \frac{[\mathrm{Tr_a}(F_2^2)]^3}{14400}\right\} \\
&\quad + (n - 496)\frac{\mathrm{tr}(R_2^6)}{725760} - \frac{Y_4 X_8}{384} \ . \qquad (12.2.27)
\end{aligned}$$

Here

$$Y_4 = \mathrm{tr}(R_2^2) - \mathrm{Tr_a}(F_2^2) \ , \qquad (12.2.28\mathrm{a})$$

$$X_8 = \mathrm{tr}(R_2^4) + \frac{[\mathrm{tr}(R_2^2)]^2}{4} - \frac{\mathrm{Tr_a}(F_2^2)\mathrm{tr}(R_2^2)}{30} + \frac{\mathrm{Tr_a}(F_2^4)}{3} - \frac{[\mathrm{Tr_a}(F_2^2)]^2}{900} \ . \qquad (12.2.28\mathrm{b})$$

Let's look up Polchinski ...

representation, for total anomaly

$$\begin{aligned}\hat{I}_{\mathrm{I}} &= \hat{I}_{\mathbf{56}}(R_2) - \hat{I}_{\mathbf{8}}(R_2) + \hat{I}_{\mathbf{8}}(F_2, R_2) \\ &= \frac{1}{1440}\left\{ -\mathrm{Tr}_{\mathrm{a}}(F_2^6) + \frac{1}{48}\mathrm{Tr}_{\mathrm{a}}(F_2^2)\mathrm{Tr}_{\mathrm{a}}(F_2^4) - \frac{[\mathrm{Tr}_{\mathrm{a}}(F_2^2)]^3}{14400} \right\} \\ &\qquad + (n-496)\left\{ \frac{\mathrm{tr}(R_2^6)}{725760} + \frac{\mathrm{tr}(R_2^4)\mathrm{tr}(R_2^2)}{552960} + \frac{[\mathrm{tr}(R_2^2)]^3}{1327104} \right\} + \frac{Y_4 X_8}{768} \,. \end{aligned} \tag{12.2.27}$$

Here

$$Y_4 = \mathrm{tr}(R_2^2) - \frac{1}{30}\mathrm{Tr}_{\mathrm{a}}(F_2^2)\,, \tag{12.2.28a}$$

$$X_8 = \mathrm{tr}(R_2^4) + \frac{[\mathrm{tr}(R_2^2)]^2}{4} - \frac{\mathrm{Tr}_{\mathrm{a}}(F_2^2)\mathrm{tr}(R_2^2)}{30} + \frac{\mathrm{Tr}_{\mathrm{a}}(F_2^4)}{3} - \frac{[\mathrm{Tr}_{\mathrm{a}}(F_2^2)]^2}{900}\,. \tag{12.2.28b}$$

Anyway, **perturbative anomaly is cancelled in 10d**.

Therefore, **it is cancelled** also in heterotic compactifications to lower dimensions, **if the internal manifold is geometric and smooth**

But the **internal CFT can be non-geometric**.

For example, in geometric compactifications to 2d, we can at most have $\mathcal{N}=(8,8)$ susy.

But there are asymmetric orbifolds which give $\mathcal{N}=(16,0)$ susy, for example.

[Melnikov-Minasian-Sethi 1707.04613]
[Florakis, García-Etxebarria, Lust, Regalado 1712.04318]

You can ask:

> ***Is perturbative anomaly automatically canceled*** *in heterotic compactifications* ***even when*** *the internal CFT is* ***non-geometric****?*

The answer is **yes**. Lerche-Nilsson-Schellekens-Warner (1988)

Their derivation is quite ingenuous.
Unfortunately I don't have time to review that.

What about the global anomalies?

In 10d $\boldsymbol{E_8 \times E_8}$, it was shown to vanish
in Witten's "Topological tools in ten dimenions" (1986).

This implies that it also vanishes in all geometric compactifications.

But again, there can be non-geometric compatifications,
where the gauge symmetry and/or charged fermions
can arise from stringy mechanism.

So, let us study them.

But this is a bit tricky because of the GS mechanism. Due to

$$dH = \mathrm{tr}\, R^2 - \mathrm{tr}\, F^2$$

the gauge transformations of B, gravity and G gauge fields mix.

Setting $F = 0$ helps, because we have one less ingredient.

So, let us study an anomaly of B and gravity,
rather than 4d $SU(2)$ anomaly, from which I started the talk.

A global anomaly in 2d

Let us consider heterotic compactification to 2d.

The worldsheet theory consists of

	c_L	c_R
(super)ghosts	-26	-15
$X^{0,1}, \bar{\psi}^{0,1}$	2	3
internal CFT	24	12

Massless fermions in the spacetime theory come from states of internal CFT where:

- The right-movers are put into R-sector,
- Massless spin-$\frac{1}{2}$ fermions are those with $(L_0, \bar{L}_0) = (1, 0)$,
- Massless gravitinos are those with $(L_0, \bar{L}_0) = (0, 0)$,
- $(-1)^{F_R}$ translates to the spacetime chirality via GSO.

The total fermion anomaly polynomial is then

$$I_4 = \Big(-24(N^+_{\text{gravitino}} - N^-_{\text{gravitino}}) + (N^+_{1/2} - N^-_{1/2})\Big)(-\frac{p_1}{48})$$

where $p_1 = \frac{1}{2}\,\mathrm{tr}(\frac{R}{2\pi})^2$.

The elliptic genus of the internal CFT is

$$\begin{aligned} Z_{\text{ell}}(q) &= \mathrm{tr}_R(-1)^{F_R} q^{L_0 - c_L/24} \bar{q}^{\bar{L_0}} \\ &= a q^{-1} + b + O(q^1) \end{aligned}$$

where

$$a = N^+_{\text{gravitino}} - N^-_{\text{gravitino}}, \qquad b = N^+_{1/2} - N^-_{1/2}.$$

String one-loop perturbation theory automatically generates the B-field one point function

$$2\pi i N \int B$$

where

$$N = \frac{1}{8\pi} \int_{\text{fund.dom. of } SL(2,\mathbb{Z})} Z_{\text{ell}}(q) \frac{dx dy}{y^2}, \qquad (\tau = x + iy)$$

which gives

$$N = -a + \frac{b}{24}.$$

[Vafa-Witten hep-th/9505053]
[Sen hep-th/9604070]

Summary so far

When the elliptic genus of the internal CFT is

$$Z_{\text{ell}}(q) = aq^{-1} + b + O(q^1),$$

the fermion anomaly is

$$I_4 = (-24a + b)(-\frac{p_1}{48})$$

while the B-field coupling is

$$2\pi i(-a + \frac{b}{24}) \int B.$$

Since

$$dH = \frac{p_1}{2},$$

the two contributions to the anomaly cancel out.

The $\boldsymbol{B}$-field coupling

$$2\pi i(-a + \frac{b}{24}) \int B$$

still poses a problem when $b/24 \notin \mathbb{Z}$,
since it transforms nontrivially under the large gauge transformation

$$B \to B + 1.$$

As b is naturally an integer, **this is a $\mathbb{Z}_{24}$ global anomaly.**

The central question can now be formulated as follows:

Take an $\mathcal{N}=(0,1)$ *SCFT with* $(c_L, c_R) = (24, 12)$.

Let's say its elliptic genus is $Z_{\text{ell}} = aq^{-1} + b + O(q^1)$.

Is b *always divisible by* 24*?*

Again I don't know how to approach this question
in the standard manner.

You can go over various 2d compactifications studied in

[Sen hep-th/9604070],

[Font, López hep-th/0405151],

[Florakis, Garcia-Etxebarria, Lust, Regalado 1712.04318].

You find that all have $\boldsymbol{b}$ divisible by $\mathbf{24}$.

How do we show this in general?

Here comes **topological modular forms** to the rescue.

But we need to recall **modular forms** first.

Modular forms

Modular forms are useful in constraining the elliptic genus of a 2d theory.

Take a 2d $\mathcal{N}=(0,1)$ theory T. Its elliptic genus is

$$Z_{\mathrm{ell}}(T;q) = \mathrm{tr}_R(-1)^{F_R} q^{L_0 - c/24} \bar{q}^{\bar{L}_0}.$$

$Z_{\mathrm{ell}}(T;q)$ is the partition function of the theory on T^2 with the periodic spin structure.

Therefore, $Z_{\mathrm{ell}}(T;q)$ is modular invariant, roughly speaking.

But this is only true up to a subtle phase which is a 24-th root of unity, due to the gravitational anomaly.

Given a 2d $\mathcal{N}=(0,1)$ theory T with $2(c_R - c_L) = \nu$,
it is useful to **add ν left-moving fermions**, to cancel the anomaly.

But the total elliptic genus is zero, due to the fermion zero modes.

We add ν fermion insertions to absorb them. What we have is then

$$\phi_W(T;q) := \eta(q)^{\nu} Z_{\text{ell}}(T;q)$$

where

$$\eta(q) = q^{1/24} \prod_n (1 - q^n).$$

(In the theory of topological modular forms,
ϕ_W is called the Witten genus.)

This combination

$$\phi_W(T;q) := \eta(q)^{\nu} Z_{\text{ell}}(T;q)$$

does not have anomalous phases (which are 24th roots of unity) under the modular transformation.

The price we paid is that

$$\phi_W(T;\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{\nu/2}\phi_W(T;\tau).$$

This transformation law defines a modular form of weight $\nu/2$.

A modular form of weight w is a function f on $\tau = x + iy$, $y > 0$, such that

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^w f(\tau).$$

Usually mathematicians require modular forms to be finite when

$$q = e^{2\pi i \tau} \to 0.$$

Then:

$$\{\text{modular forms}\} = \mathbb{C}[E_4, E_6]$$

where

$$E_k = \sum_{(n,m)\neq 0} \frac{1}{(n\tau + m)^k}$$

is the Eisenstein series of weight k.

Our $\phi_W(T;q)$ can have poles at $q \to 0$. For this purpose we use

$$\Delta(q) := \eta(q)^{24} = q + O(q^2).$$

So

$$\left\{ \begin{array}{c} \text{modular forms} \\ \text{allowing poles at } q \end{array} \right\} = \mathbb{C}[E_4, E_6, \Delta^{-1}].$$

Our $\phi_W(T;q)$ has integer coefficients in q-expansions.

It is known that

$$\left\{ \begin{array}{c} \text{modular forms with} \\ \text{integer coefficients} \end{array} \right\} = \frac{\mathbb{Z}[c_4, c_6, \Delta]}{c_4^3 - c_6^2 = 1728\Delta}$$

where

$$c_4 = \frac{45}{\pi^4} E_4 = 1 + 240q^2 + \cdots , \quad c_6 = \frac{945}{2\pi^6} E_6 = 1 - 504q^2 + \cdots ,$$

so

$$\left\{ \begin{array}{c} \text{modular forms with} \\ \text{integer coefficients and poles} \end{array} \right\} = \frac{\mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}]}{c_4^3 - c_6^2 = 1728\Delta}.$$

We come back to our original question:

Take an $\mathcal{N}{=}(0,1)$ *SCFT with* $(c_L, c_R) = (24, 12)$.

Let's say its elliptic genus is $Z_{\text{ell}} = aq^{-1} + b + O(q^1)$.

Is b *always divisible by* 24*?*

Since $\nu = 2(c_R - c_L) = -24$, we consider

$$\phi_W(T;q) = \eta(q)^{-24} Z_{\text{ell}}(T;q) = \Delta^{-1}(aq^{-1} + b + \cdots).$$

This is a modular form of weight -12 with integer coefficients and poles of order at most 2. From this we conclude

$$\phi_W(T;q) = ac_4^3\Delta^{-2} + (-744a + b)\Delta^{-1}$$

$$\in \frac{\mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}]}{c_4^3 - c_6^2 = 1728\Delta}.$$

This alone does not tell that b is divisible by 24.

Topological modular forms

Topological modular forms $\boldsymbol{TMF}$ are
a **topological version of modular forms**.

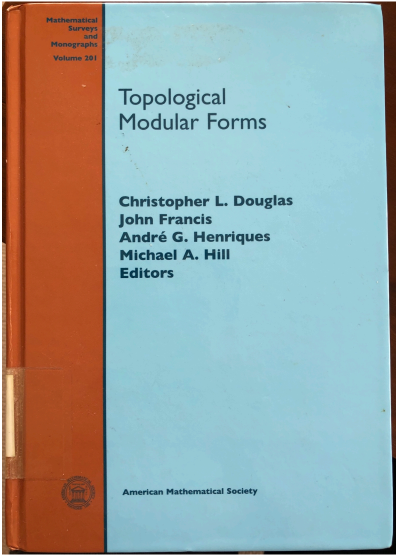
Distantly inspired by Witten's work on elliptic genus in 1986.

$\boldsymbol{TMF}$ was mathematically constructed by Hopkins et al., around 2000, using an amalgam of algebraic topology and algebraic geometry.

[e.g. Hopkins math.AT/0212397]

Far above my head.

There's a textbook I've been reading these days...



https://doi.org/10.1090/surv/201

https://mathscinet.ams.org/mathscinet-getitem?mr=3223024

For us the Segal-Stolz-Teichner conjecture helps:

$$\boldsymbol{TMF_\nu} = \frac{\left\{ \begin{array}{c} \text{2d } \boldsymbol{\mathcal{N}=(0,1)} \text{ supersymmetric theory} \\ \text{with } \boldsymbol{\nu = 2(c_R - c_L)} \end{array} \right\}}{\text{continuous deformation}}$$

[Segal 1988]
[Stolz-Teichner 2002]
[Stolz-Teichner 1108.0189]

There is a mathematically well-defined map

$$\phi_W : TMF_\nu \to \left\{ \begin{array}{c} \text{modular forms with} \\ \text{integer coefficients and poles} \end{array} \right\}_{\text{of weight } \nu/2}$$

$$= \left(\frac{\mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}]}{c_4^3 - c_6^2 = 1728\Delta} \right)_{\text{of weight } \nu/2}.$$

Under the conjecture

$$TMF_\nu = \frac{\left\{ \begin{array}{c} \text{2d } \mathcal{N}{=}(0,1) \text{ supersymmetric theory} \\ \text{with } \nu = 2(c_R - c_L) \end{array} \right\}}{\text{continuous deformation}}$$

this map computes

$$\phi_W(T; q) = \eta(q)^\nu Z_{\text{ell}}(T; q).$$

The image of the map

$$\phi_W : TMF_\nu \to \left\{ \begin{array}{c}\text{modular forms with}\\ \text{integer coefficients and poles}\end{array} \right\}_{\text{of weight } \nu/2}$$

$$= \left(\frac{\mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}]}{c_4^3 - c_6^2 = 1728\Delta} \right)_{\text{of weight } \nu/2}.$$

was mathematically determined. It is generated by

$$a_{i,j,k} {c_4}^i {c_6}^j \Delta^k, \qquad (i \geq 0;\ j = 0, 1;\ k \in \mathbb{Z})$$

where

$$a_{i,j,k} = \begin{cases} 24/\gcd(24, k) & (i = j = 0), \\ 2 & (j = 1), \\ 1 & (\text{otherwise}). \end{cases}$$

Combined with Segal-Stolz-Teichner conjecture, this means that

We know exactly which modular function appears as the elliptic genus of 2d $\mathcal{N}=(\mathbf{0},\mathbf{1})$ *supersymmetric theories.*

We can finally come back to our question

Take an $\mathcal{N}=(0,1)$ *SCFT with* $(c_L, c_R) = (24, 12)$.

Let's say its elliptic genus is $Z_{\text{ell}} = aq^{-1} + b + O(q^1)$.

Is b *always divisible by* 24*?*

Recall that otherwise heterotic compactifications to 2d has a $\mathbb{Z}_{24}$ global anomaly.

We already argued that

$$\phi_W(T;q) = ac_4^3\Delta^{-2} + (-744a + b)\Delta^{-1}$$

but this should be in the image of

$$\phi_W : TMF_\nu \to \{ \begin{matrix}\text{modular forms with} \\ \text{integer coefficients and poles}\end{matrix} \}_{\text{of weight } \nu/2}$$

$$= \left(\frac{\mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}]}{c_4^3 - c_6^2 = 1728\Delta} \right)_{\text{of weight } \nu/2}.$$

So

$$\phi_W(T;q) = ac_4^3\Delta^{-2} + (-744a + b)\Delta^{-1}$$

should be an integral linear combination of

$$a_{i,j,k}{c_4}^i{c_6}^j\Delta^k, \qquad (i \geq 0;\ j = 0, 1;\ k \in \mathbb{Z})$$

where

$$a_{i,j,k} = \begin{cases} 24/\gcd(24, k) & (i = j = 0), \\ 2 & (j = 1), \\ 1 & (\text{otherwise}). \end{cases}$$

Taking $k = -1$, we find that $-744a + b$ is a multiple of

$$24/\gcd(24, -1) = 24.$$

So b is divisible by 24. **Done.**

Comments

Clearly we are not really done.

What I did was to transfer

the question of global anomalies of heterotic strings

to

the validity of the Segal-Stolz-Teichner conjecture

$$\boldsymbol{TMF_\nu} = \frac{\left\{ \begin{array}{c} \text{2d } \boldsymbol{\mathcal{N}{=}(0,1)} \text{ supersymmetric theory} \\ \text{with } \boldsymbol{\nu = 2(c_R - c_L)} \end{array} \right\}}{\text{continuous deformation}}$$

Two questions immediately arise:

- Assuming the conjecture, what can we say?
- How about the conjecture itself?

Assuming the conjecture, what can we say?

Today I only discussed 2d $\mathbb{Z}_{24}$ anomaly.

How about the 4d $SU(2)$ anomaly I started the talk?

How about other global anomalies in heterotic string theories?

Recall:

> *Take an* $SU(2)$*-symmetric* $\mathcal{N}=(0,1)$ *SCFT with* $(c_L, c_R) = (22, 9)$.
>
> *Let* N_k *be the number of states in the* k*-dim'l irrep of* $SU(2)$ *with* $(L_0, \bar{L}_0) = (1, 0)$ *in the R-sector.*
>
> *Is* $(-1)^{N_2+N_6+N_{10}+\cdots} = +1$ *?*

Such a theory determines a class

$$T \in TMF_\nu(BSU(2))_k$$

where $\nu = 2(9 - 22) = -26$
and k is the level of the $SU(2)$ current algebra.

The absence of Witten anomaly can then be translated to a certain property of

$$\boldsymbol{TMF_{-26}(BSU(2))_k.}$$

Unfortunately nobody has computed this group, and I found nobody who could compute it for me.

More generally, the question of global anomaly of heterotic compactifications down to d dimensions with gauge symmetry G can be translated to the study of

$$TMF_{-22-d}(BG)_k.$$

Each group is very hard to compute.

More generally, the question of global anomaly of heterotic compactifications down to d dimensions with gauge symmetry G can be translated to the study of

$$TMF_{-22-d}(BG)_k.$$

Each group is very hard to compute.

But **there is a way** to show that global anomaly always vanishes, by **considering all cases at once**, **without doing any case-by-case analyses**.

[Work in progress with Yamashita, a mathematician in Kyoto]

How about the conjecture itself?

$$TMF_\nu = \frac{\left\{ \begin{array}{c} \text{2d } \mathcal{N}=(0,1) \text{ supersymmetric theory} \\ \text{with } \nu = 2(c_R - c_L) \end{array} \right\}}{\text{continuous deformation}}$$

It will be very hard to get a mathematically rigorous proof.
The RHS isn't even defined yet!

Many subtle properties on the LHS are known.

They translate to **many subtle properties of 2d theories** which are not at all apparent to us.

One example is $TMF_\nu \simeq TMF_{\nu+576}$. Why?

As another example, let us take $TMF_3 = \mathbb{Z}_{24}$.

This means that 2d $\mathcal{N}{=}(0,1)$ theories with $2(c_R - c_L) = 3$ can be classified by $\mathbb{Z}_{24}$.

Examples in each class $k \in \mathbb{Z}_{24}$ are believed to be given by

$\mathcal{N}{=}(0,1)$ sigma models on $S^3 = SU(2)$ with WZW level k.

How to see the mod-24 behavior in k was discussed in [Gaiotto, Johnson-Freyd, Witten 1902.10249].

How to extract a $\mathbb{Z}_{24}$ invariant from such a theory was discussed in [Gaiotto, Johnson-Freyd 1904.05788].

Recall also that the image of the map

$$\phi_W : TMF_\nu \to \left\{ \begin{array}{c} \text{modular forms with} \\ \text{integer coefficients and poles} \end{array} \right\}_{\text{of weight } \nu/2}$$

$$= \left(\frac{\mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}]}{c_4^3 - c_6^2 = 1728\Delta} \right)_{\text{of weight } \nu/2}.$$

was mathematically determined. It is generated by

$$a_{i,j,k} {c_4}^i {c_6}^j \Delta^k, \qquad (i \geq 0;\ j = 0, 1;\ k \in \mathbb{Z})$$

where

$$a_{i,j,k} = \begin{cases} 24/\gcd(24, k) & (i = j = 0), \\ 2 & (j = 1), \\ 1 & (\text{otherwise}). \end{cases}$$

One consequence in our language is as follows.

Consider a 2d $\mathcal{N}=(0,1)$ *theory with* $2(c_R - c_L) = 24k$.

If its elliptic genus is constant, it is a multiple of $24/\gcd(24,k)$.

Such theories for $1 \le k \le 5$ were constructed
in [Gaiotto, Johnson-Freyd 1811.00589].

In particular,

> *If the elliptic genus of 2d* $\mathcal{N}{=}(0,1)$ *theory is simply* 1*,*
> *then* $c_L - c_R$ *is divisible by* 288*.*
>
> *Conversely, there should be a 2d* $\mathcal{N}{=}(0,1)$ *theory*
> *whose elliptic genus is* 1 *and* $c_L - c_R = \pm 288$*.*

This is an open question and I consider it quite important.

Summary

Today, I considered **global anomalies in heterotic string theories.**

Such questions can be answered using the **mathematical theory of TMF**, using the **Segal-Stolz-Teichner conjecture**:

$$TMF_\nu = \frac{\left\{ \begin{array}{c} \text{2d } \mathcal{N}{=}(0,1) \text{ supersymmetric theory} \\ \text{with } \nu = 2(c_R - c_L) \end{array} \right\}}{\text{continuous deformation}}$$

This conjecture predicts **many unexplored properties of 2d theories**, which I think are worth pursuing.