QFT perspectives on topological modular forms

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I've been working on **topological modular forms** (**TMF**)
for about three years, since the start of the COVID pandemic.

TMF is an esoteric subject in algebraic topology / homotopy theory.

But **it is also thought to classify 2d $\boldsymbol{\mathcal{N}=(0,1)}$ SQFTs.**

It also has applications in the study of heterotic strings.

Brief timeline:

Jan 2020	started learning about **TMF**	
Jun 2021	gave a talk in this seminar series	[slides] [video]
Sep 2023	giving a talk here again	

I feel I understand **TMF** much better now!

(If you have downloaded the PDF file of this talk,
[purple texts] are hyperlinked.)

I'd like to give an introduction to this fascinating subject.

I'd like to start with the baby version,
and then proceed to the 'real' version.

	obj. in alg. top.	classifies
baby version	$\mathbf{K}$	SQM = 1d SQFT
'real' version	$\mathbf{TMF}$	2d SQFT

SQM and K-theory

Let me start with the relation between

supersymmetric quantum mechanics

and

K-theory.

Suppose you really like SQM.

You'd like to classify SQM. How should you proceed?

Mimimal ingredients are $(-1)^F$ and $Q = Q^\dagger$ satisfying

$$\{(-1)^F, Q\} = 0.$$

The Hamiltonian is $H = Q^2$.

Take an eigenstate $|v\rangle$ of H with eigenvalue E:

$$E = \langle v|H|v\rangle = \langle v|QQ|v\rangle = \left\| Q\,|v\rangle \right\|^2 \geq 0$$

So we have

$$|+\rangle \xleftrightarrow{Q} |-\rangle \qquad (E > 0)$$

while

$$\begin{aligned} |+\rangle &\xrightarrow{Q} 0, \\ |-\rangle &\xrightarrow{Q} 0, \end{aligned} \qquad (E = 0).$$

The Witten index Z is defined to be $Z = \mathrm{tr}(-1)^F e^{-\beta H}$.

Equivalently, it is

the number of zero energy states with $(-1)^F = +1$
minus the number of zero energy states with $(-1)^F = -1$.

This is independent of continuous deformation of the system in question:

$$\begin{tikzcd}
|+\rangle \arrow[rr, leftrightarrow, "Q"] & {} \arrow[d, Leftrightarrow] & |-\rangle & (E > 0) \\
|+\rangle \arrow[r, "Q"] & 0 & |-\rangle \arrow[l, "Q"'] & (E = 0)
\end{tikzcd}$$

Are two SQM with the same Witten index continuously connected?

Clearly there are counterexamples. For example:

1. A system with a single state $|+, E = 0\rangle$
2. A system with a $|+, E = 0\rangle$ and a pair $|\pm, E > 0\rangle$.

both have Witten index $= 1$.

But you can't change the dimension of the Hilbert space continuously!

Note that a system with a single pair $|\pm, E > 0\rangle$ breaks supersymmetry.

Let's declare that **two systems are equivalent when** they are connected via **(a) continuous deformation** and

(b) addition/removal of susy-breaking sector.

Then two systems

1. A system with a single state $|+, \boldsymbol{E = 0}\rangle$
2. A system with a $|+, \boldsymbol{E = 0}\rangle$ and a pair $|\pm, \boldsymbol{E > 0}\rangle$.

are equivalent.

It is easy to see that the Witten index is a complete invariant,
i.e. two systems are equivalent iff the Witten indices are the same.

Let's spice it up.
Consider **two** Majorana fermion operators $\psi_{1,2} = \psi_{1,2}^\dagger$.

It has a standard two-dimensional irreducible representation

$$\psi_1 = \sigma_1, \quad \psi_2 = \sigma_2, \quad (-1)^F = \sigma_3.$$

Then $m = 2n$ Majorana fermions are irreducibly represented on a Hilbert space with $2^{m/2} = 2^n$ states.

What happens with a **single** Majorana fermion, i.e. when $m = 1$?
There's no Hilbert space whose dimension is $2^{1/2}$.

It's a type of 'gravitational' anomaly. An anomalous system in this sense requires an additional $\psi = \psi^\dagger$ to be irreducibly quantized.

(See Sec.2.1 of [Witten 2305.01012] for recent pedagogical explanation.)

Let's then classify SQM with this gravitational anomaly.

The ingredients are

$$\{(-1)^F, Q\} = 0$$

together with an additional ψ satisfying

$$\{(-1)^F, \psi\} = \{Q, \psi\} = 0.$$

How is the classification affected?

Well, a minimal susy-preserving example is

$$(-1)^F = \sigma_3, \quad \psi = \sigma_1, \quad Q = 0$$

but it can be continuously connected to susy-breaking choice

$$(-1)^F = \sigma_3, \quad \psi = \sigma_1, \quad Q = c\sigma_2.$$

So every SQM with this anomaly can be trivialized.

Classification of SQM up to
(a) continuous deformation and
(b) addition/removal of susy-breaking sector
is given by

grav. anomaly	0	1
classification	$\mathbb{Z}$	0

The complete invariant is given by the Witten index.

Let's spice it up further!

Suppose you're interested in **time-reversal-invariant SQM**.

Time-reversal is given by an anti-linear T with $T^2 = 1$ with

$$[T, (-1)^F] = [T, Q] = 0.$$

Gravitational anomaly in this case is mod 8 rather than mod 2, again carried by Majorana fermions.

(See Sec.3.2 of [Witten 2305.01012] for recent pedagogical explanation.)

So the ingredients are: $(-1)^F$, Q, T and $\psi_{1,\ldots,n}$ with $[T, \psi_i] = 0$.

Classification of time-reversal-invariant SQM up to

(a) continuous deformation and

(b) addition/removal of susy-breaking sector

is known to be given by

grav. anomaly	0	1	2	3	4	5	6	7
classification	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

The cases with grav. anomaly $= \mathbf{0}$ and $= \mathbf{4}$ are distinguished by ordinary Witten index.

The cases with grav. anomaly $= \mathbf{1}$ and $= \mathbf{2}$ are distinguished by the mod-2 Witten index.

What is the mod-2 Witten index?

Consider the case when the grav. anomaly $= \mathbf{1}$:

grav. anomaly	0	**1**	2	3	4	5	6	7
classification	$\mathbb{Z}$	$\mathbb{Z}_{\mathbf{2}}$	$\mathbb{Z}_{\mathbf{2}}$	0	$\mathbb{Z}$	0	0	0

Recall that the ingredients are

$$\boldsymbol{(-1)^{F}, Q, T, \psi.}$$

When $\boldsymbol{H = 0}$ the irrep has the structure

$$|+\rangle \xleftrightarrow{\psi} |-\rangle$$

while when $\boldsymbol{H > 0}$ the irrep looks like

$$\begin{tikzcd}
{|+\rangle} \arrow[r, leftrightarrow, "\psi"] \arrow[d, leftrightarrow, "Q"'] & {|-\rangle} \arrow[d, leftrightarrow, "Q"] \\
{|-'\rangle} \arrow[r, leftrightarrow, "\psi"] & {|+'\rangle}
\end{tikzcd}$$

where every basis state is $\boldsymbol{T}$ invariant. So

$$\frac{\mathbf{1}}{\mathbf{2}}(\text{\# of zero energy states})$$

is a mod-2 invariant.

So, the classification of SQM without time-reversal is

grav. anomaly	0	1
classification	$\mathbb{Z}$	0

and the same with time reversal is

grav. anomaly	0	1	2	3	4	5	6	7
classification	$\mathbb{Z}$	$\mathbb{Z}_{\mathbf{2}}$	$\mathbb{Z}_{\mathbf{2}}$	0	$\mathbb{Z}$	0	0	0

These exactly matches with the math results

$\boldsymbol{n}$	0	1
$\mathbf{K}_{\boldsymbol{n}}$	$\mathbb{Z}$	0

and

$\boldsymbol{n}$	0	1	2	3	4	5	6	7
$\mathbf{KO}_{\boldsymbol{n}}$	$\mathbb{Z}$	$\mathbb{Z}_{\mathbf{2}}$	$\mathbb{Z}_{\mathbf{2}}$	0	$\mathbb{Z}$	0	0	0

Why?

Well, K theory and KO theory have many different definitions leading to the same generalized (co)homology theory.

One definition in [Atiyah-Singer 1969] happens to literally agree with what we've been doing so far.

Somehow it long predates Witten's introduction of SQM ...

(For a review, see Appendix of [YT-Yamashita-Yonekura 2302.07548].)

2d SQFT and TMF

Part 1: Ordinary and mod-2 elliptic genera

We can summarize our discussion so far as follows:

$$\mathbf{KO}_{\boldsymbol{n}} = \frac{\left\{ \begin{array}{c} \text{Time-reversal-invariant SQM} \\ \text{with grav. anomaly } \boldsymbol{n} \in \mathbb{Z}_{\mathbf{8}} \end{array} \right\}}{\begin{array}{c} \text{continuous deformation and/or} \\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector} \end{array}}$$

Satisfied with the understanding of SQM = 1d SQFT,
we'd like to do the same with 2d SQFT.

Take the minimal amount of supersymmetry, $\boldsymbol{\mathcal{N}=(0,1)}$.

So we'd like to classify 2d $\boldsymbol{\mathcal{N}=(0,1)}$ theories up to

(a) continuous deformation and
(b) addition/removal of susy-breaking sector.

My convention is to put SUSY on the right-moving sector.

In SQM = 1d SQFT, the grav. anomaly was carried
e.g. by Majorana fermions.

Also in 2d, the grav. anomaly is carried e.g. by Majorana fermions, characterized by the anomaly polynomial

$$n\frac{p_1}{48}, \qquad n \in \mathbb{Z}.$$

For CFTs, it's given by

$$n = 2(c_R - c_L),$$

but n makes sense even for non-conformal theories.

So our question is

$$??? = \frac{\left\{\begin{array}{c}\text{2d } \boldsymbol{\mathcal{N}}=(\mathbf{0},\mathbf{1}) \text{ SQFT}\\ \text{with grav. anomaly } \boldsymbol{n}\in\mathbb{Z}\end{array}\right\}}{\begin{array}{c}\text{continuous deformation and/or}\\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector}\end{array}}$$

.

Recall

$$\mathbf{KO}_{n} = \frac{\left\{ \begin{matrix} \text{Time-reversal-invariant SQM} \\ \text{with grav. anomaly } \boldsymbol{n} \in \mathbb{Z}_{\mathbf{8}} \end{matrix} \right\}}{\begin{matrix} \text{continuous deformation and/or} \\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector} \end{matrix}}$$

(Here I′m using the time-reversal-invariant version in SQM = 1d SQFT, because 2d theories automatically come with CPT.)

Mathematicians say that the (co)homology theories

$$\mathbf{H}_n(-), \qquad \mathbf{KO}_n(-), \qquad \mathbf{TMF}_n(-)$$

form a natural progression, where $\mathbf{TMF}_n$ is the **topological modular forms**.

Very roughly, $\mathbf{H}_n$ is 0d SQFT = theory of ordinary differential forms.

We saw that $\mathbf{KO}_n$ captures 1d SQFT = SQM.

Then it's likely that we have the following statement:

$$\mathbf{TMF}_{\boldsymbol{n}} = \frac{\left\{\begin{array}{c}\text{2d } \boldsymbol{\mathcal{N}}{=}\mathbf{(0,1)} \text{ SQFT} \\ \text{with grav. anomaly } \boldsymbol{n} \in \mathbb{Z}\end{array}\right\}}{\begin{array}{c}\text{continuous deformation and/or} \\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector}\end{array}}$$

This is the conjecture of Segal-Stolz-Teichner.

[Segal 1988] [Stolz-Teichner 2002] [Stolz-Teichner 1108.0189]

There are more and more physics pieces of evidence since 2018.

Before discussing mathematical properties of $\mathbf{TMF}_{n}$, let's study

$$\frac{\left\{\begin{array}{c}\text{2d } \boldsymbol{\mathcal{N}=(0,1)} \text{ SQFT}\\ \text{with grav. anomaly } \boldsymbol{n}\in\mathbb{Z}\end{array}\right\}}{\begin{array}{c}\text{continuous deformation and/or}\\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector}\end{array}}$$

from physics points of view.

Question:

How do we detect such equivalence classes?

General answer:

Find functions

$$f : \{\text{SQFTs}\} \to \text{numbers}$$

which are invariant under deformations.

This is exactly what we did in the case of SQM,
for which we used ordinary and mod-2 Witten indices.

Classic example:

Elliptic genus

[Witten 1989]

- **the generating function of the Witten index** of the system on R-sector S^1 for each value of the momenta P around S^1:

$$Z(q) = \mathrm{tr}_{\mathcal{H}^R_{S^1}} (-1)^F q^{L_0 - c_L/24} \bar{q}^{\bar{L}_0 - c_R/24}$$
$$= \sum_P (\text{Witten index at fixed } P)\, q^P$$

- nonzero only when $n = 2(c_R - c_L) \equiv 0$ mod 4.
- is a Laurent polynomial with coefficients in $\mathbb{Z}$.
- is a **modular form**.

Another example:

Mod-2 elliptic genus [YT-Yamashita-Yonekura 2302.07548]

- **the generating function of the mod-2 Witten index** of the system on R-sector S^1 for each value of the momenta P around S^1:

$$Z(q) = \sum_P (\text{mod-2 Witten index at fixed } P)\, q^P$$

- nonzero only when $n = 2(c_R - c_L) \equiv 1, 2 \bmod 8$.
- is a Laurent polynomial with coefficients in $\mathbb{Z}_2$.
- is a **mod-2 modular form**.

In essence, physics provides

$$\frac{\left\{\begin{array}{c}\text{2d } \boldsymbol{\mathcal{N}=(0,1)} \text{ SQFT}\\ \text{with grav. anomaly } \boldsymbol{n} \in \mathbb{Z}\end{array}\right\}}{\begin{array}{c}\text{continuous deformation and/or}\\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector}\end{array}} \longrightarrow \left\{\begin{array}{c}\text{modular forms with}\\ \text{coefficients in } \mathbf{KO}_{\boldsymbol{n}}\end{array}\right\}.$$

As we have

$\boldsymbol{n}$	0	1	2	3	4	5	6	7
$\mathbf{KO}_{\boldsymbol{n}}$	$\mathbb{Z}$	$\mathbb{Z}_{\mathbf{2}}$	$\mathbb{Z}_{\mathbf{2}}$	0	$\mathbb{Z}$	0	0	0

this provides ordinary elliptic genus when $\boldsymbol{n} \equiv \mathbf{0}, \mathbf{4}$ mod 8 and mod-2 elliptic genus when $\boldsymbol{n} \equiv \mathbf{1}, \mathbf{2}$ mod 8.

A large source of 2d $\mathcal{N}=(0,1)$ SQFTs are the $\mathcal{N}=(0,1)$ sigma models on a manifold M_d (where d denotes the dimension).

Each coordinate X^i comes with a right-moving superpartner ψ^i_R.

We need to have a B-field satisfying

$$dH = \frac{1}{2}p_1(R)$$

on M_d, for the cancellation of worldsheet anomaly.

Such sigma models have $n = 2(c_R - c_L) = d$.

So we have

$$\left\{ \begin{array}{c} \boldsymbol{M_n} \text{ with} \\ \boldsymbol{B}\text{-field} \end{array} \right\} \to \frac{\left\{ \begin{array}{c} \text{2d } \boldsymbol{\mathcal{N}}\boldsymbol{=(0,1)} \text{ SQFT} \\ \text{with grav. anomaly } \boldsymbol{n} \in \mathbb{Z} \end{array} \right\}}{\begin{array}{c} \text{continuous deformation and/or} \\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector} \end{array}} \to \left\{ \begin{array}{c} \text{modular forms with} \\ \text{coefficients in } \mathbf{KO}_{\boldsymbol{n}} \end{array} \right\}$$

Math also provides

$$\left\{ \begin{array}{c} \boldsymbol{M_n} \text{ with} \\ \boldsymbol{B}\text{-field} \end{array} \right\} \to \mathbf{TMF}_{\boldsymbol{n}} \to \left\{ \begin{array}{c} \text{modular forms with} \\ \text{coefficients in } \mathbf{KO}_{\boldsymbol{n}} \end{array} \right\}.$$

Note only that, the following diagram commutes:

$$\begin{tikzcd}
 & \dfrac{\left\{\begin{array}{c}\text{2d } \boldsymbol{\mathcal{N}=(0,1)} \text{ SQFT}\\ \text{with grav. anomaly } \boldsymbol{n} \in \mathbb{Z}\end{array}\right\}}{\begin{array}{c}\text{continuous deformation and/or}\\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector}\end{array}} \arrow[dr] & \\
\left\{\begin{array}{c}\boldsymbol{M_n} \text{ with}\\ \boldsymbol{B}\text{-field}\end{array}\right\} \arrow[ur] \arrow[dr] & & \left\{\begin{array}{l}\text{modular forms with}\\ \text{coefficients in } \mathbf{KO}_{\boldsymbol{n}}\end{array}\right\} \\
 & \mathbf{TMF}_{\boldsymbol{n}} \arrow[ur] &
\end{tikzcd}$$

where the upper and lower paths can be studied by physicists and mathematicians, respectively.

This is a strong piece of supporting evidence of

$$\mathbf{TMF}_{n} = \frac{\left\{\begin{array}{c}\text{2d } \mathcal{N}=(0,1) \text{ SQFT}\\ \text{with grav. anomaly } n \in \mathbb{Z}\end{array}\right\}}{\begin{array}{c}\text{continuous deformation and/or}\\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector}\end{array}},$$

the Segal-Stolz-Teichner conjecture.

2d SQFT and TMF

Part 2: Bunke-Nauman invariant

Question:

Do ordinary and mod-2 elliptic genus characterize

$$\mathbf{TMF}_{n} \sim \frac{\left\{ \begin{array}{c} \text{2d } \mathcal{N}{=}(\mathbf{0},\mathbf{1}) \text{ SQFT} \\ \text{with grav. anomaly } \boldsymbol{n} \in \mathbb{Z} \end{array} \right\}}{\begin{array}{c} \text{continuous deformation and/or} \\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector} \end{array}} \, ?$$

Answer:

No.

[Bunke and Naumann 0912.4875]
[Berwick-Evans 1510.06464] constructed a subtler invariant

$$\mathbf{TMF}_n \to \frac{\mathbb{R}((q))}{\mathbb{Z}((q)) + \text{modular forms}}$$

when $n = 3$ or 7 mod 8.

Here $\mathbb{X}((q))$ is the ring of Laurent series in q with $\mathbb{X}$ coefficients.

(Note that ordinary and mod-2 elliptic genera are nonzero only for $n = 0, 1, 2, 4$ mod 8, so they vanish for $n = 3, 7$ mod 8.)

[Gaiotto and Johnson-Freyd 1904.05788]
[Yonekura 2207.13858] gave the physics version:

$$\frac{\left\{\begin{array}{c}\text{2d } \mathcal{N}=(\mathbf{0},\mathbf{1}) \text{ SQFT}\\ \text{with grav. anomaly } \boldsymbol{n}\in\mathbb{Z}\end{array}\right\}}{\begin{array}{c}\text{continuous deformation and/or}\\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector}\end{array}} \to \frac{\mathbb{R}((\boldsymbol{q}))}{\mathbb{Z}((\boldsymbol{q})) + \text{modular forms}}$$

again when $\boldsymbol{n} = \mathbf{3}$ or $\mathbf{7}$ mod 8.

The approach of Gaiotto and Johnson-Freyd is to consider a mock modular form associated to a given theory. This gives an invariant which characterize the failure of this mock modular form to be truly modular, explaining the RHS.

The following diagram is expected to commute:

$$\begin{tikzcd}
 & \dfrac{\left\{\begin{array}{c}\text{2d } \boldsymbol{\mathcal{N}=(0,1)} \text{ SQFT}\\ \text{with grav. anomaly } \boldsymbol{n} \in \mathbb{Z}\end{array}\right\}}{\begin{array}{c}\text{continuous deformation and/or}\\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector}\end{array}} \arrow[dr] & \\
\left\{\begin{array}{c}\boldsymbol{M_n} \text{ with}\\ \boldsymbol{B}\text{-field}\end{array}\right\} \arrow[ur] \arrow[dr] & & \dfrac{\mathbb{R}((\boldsymbol{q}))}{\mathbb{Z}((\boldsymbol{q})) + \text{modular forms}} \\
 & \mathbf{TMF}_{\boldsymbol{n}} \arrow[ur] &
\end{tikzcd}$$

where $\boldsymbol{n} = \mathbf{3}$ or $\mathbf{7}$ mod 8.

This is called the Bunke-Naumann invariant.

For example, for the $\mathcal{N}=(0,1)$ sigma model on S^3 with $\int_{S^3} H = k$, or equivalently the $\mathcal{N}=(0,1)$ WZW model on $SU(2)$ at level k.

The Bunke-Naumann invariant turns out to be

$$\frac{k}{24} \in \mathbb{R}/\mathbb{Z},$$

both mathematically and physically.

It is consistent with the existence of an explicit deformation of the $k = 24$ model to null [Gaiotto, Johnson-Freyd, Witten 1902.10249].

2d SQFT and TMF

Part 3: Even subtler parts

Question:

Does the combination of **ordinary or mod-2 elliptic genus** and **Bunke-Naumann invariant** completely detect

$$\mathbf{TMF}_n \sim \frac{\left\{\begin{array}{c}\text{2d } \mathcal{N}=(0,1) \text{ SQFT} \\ \text{with grav. anomaly } n \in \mathbb{Z}\end{array}\right\}}{\begin{array}{c}\text{continuous deformation and/or} \\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector}\end{array}} \ ?$$

Answer:

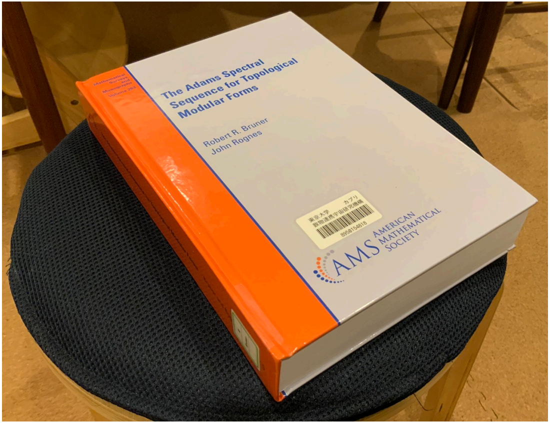
No!

Let A_n be the subgroup of TMF_n
whose ordinary/mod-2 elliptic genus is zero.

These are the truly interesting part of TMF_n!

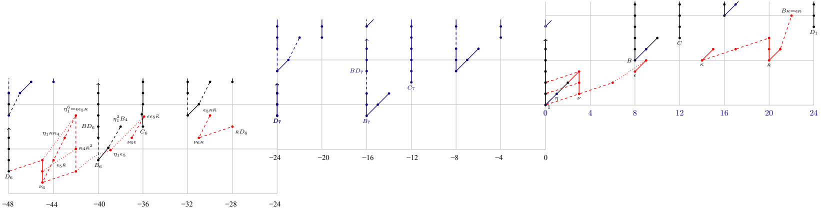
How do we know these?

The latest standard reference on $\mathbf{TMF}_n$ is [Bruner-Rognes 2021]



This has about 700 pages, but is not a textbook ;
it just documents the computation of $\mathbf{TMF}_n$ in detail.

A table from this book looks like this



Here the horizontal axis is $\boldsymbol{n}$, a dot is $\mathbb{Z}_\mathbf{2}$,
when $\boldsymbol{n}$ dots are connected vertically they mean $\mathbb{Z}_{\mathbf{2}^{\boldsymbol{n}}}$,
when $\boldsymbol{\infty}$ dots are done so they mean $\mathbb{Z}$, etc.

Black dots have nonzero ordinary or mod-2 Witten indices,
and **red dots** are the most interesting ones.

According to them, in the range $-31 \leq n \leq 9$,
the nonzero cases are:

$$\begin{aligned} &A_3 = \mathbb{Z}_{24}, \quad A_6 &&= \mathbb{Z}_2, \quad A_8 &&= \mathbb{Z}_2, \quad A_9 &&= \mathbb{Z}_2, \ldots \\ & \quad A_{-28} &&= \mathbb{Z}_2, \quad A_{-30} &&= \mathbb{Z}_2, \quad A_{-31} &&= \mathbb{Z}_2, \ldots \end{aligned}$$

$A_3 = \mathbb{Z}_{24}$ is detected by Bunke-Naumann invariant,
but what are the others?

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$A_3 = \mathbb{Z}_{24}$ is detected by Bunke-Naumann invariant,
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$A_{3,6,8,9}$ are $\mathcal{N}{=}(0,1)$ WZW models on

$$SU(2) \qquad SU(2)^2 \qquad SU(3) \qquad SU(2)^3$$

[Hopkins math.AT/0212397]

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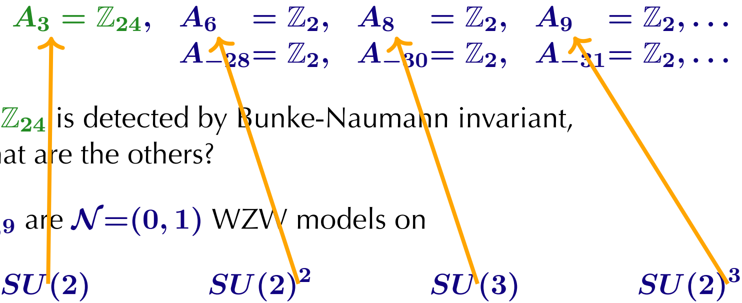

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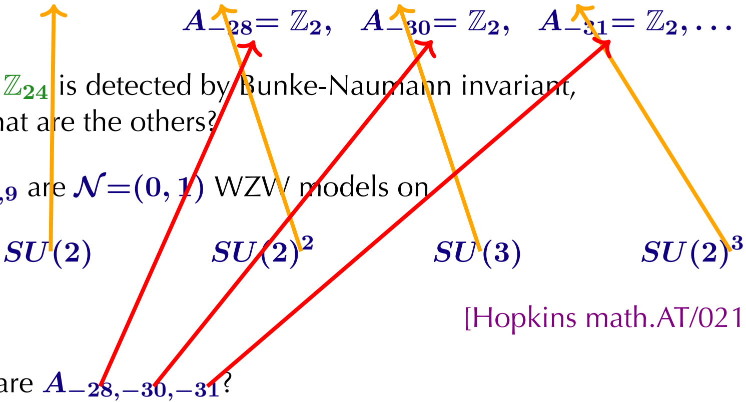

$$\begin{array}{llll} A_3=\mathbb{Z}_{24}, & A_6 = \mathbb{Z}_2, & A_8=\mathbb{Z}_2, & A_9=\mathbb{Z}_2,\dots\\ & A_{-28}=\mathbb{Z}_2, & A_{-30}=\mathbb{Z}_2, & A_{-31}=\mathbb{Z}_2,\dots\end{array}$$

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$A_{3,6,8,9}$ are $\mathcal{N}{=}(0,1)$ WZW models on

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[Hopkins math.AT/0212397]

What are $A_{-28,-30,-31}$?

Here the classification of spin holomorphic CFTs comes in.

Stolz-Teichner conjecture concerns $\mathcal{N}=(0,1)$ SQFTs
and $n = 2(c_R - c_L)$.

Purely left-moving (i.e. $c_L > 0$, $c_R = 0$) **non-supersymmetric** modular-invariant spin **CFTs are actually** $\mathcal{N}=(0,1)$ **SQFTs with** $n = -2c_L$.

These are classified recently in

[Boyle Smith, Lin, YT, Zheng 2303.16917]	($c_L \leq 16$)
[Rayhaun 2303.16921]	($c_L \leq 24$)
[Höhn-Möller 2303.17190]	($c_L \leq 24$)

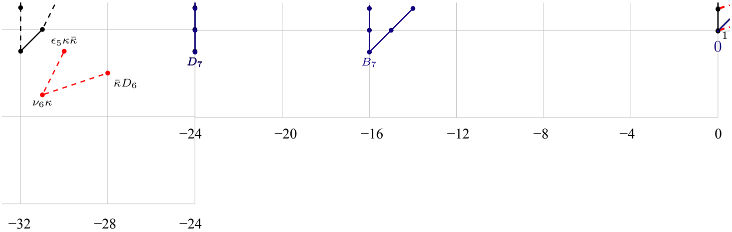
The irreducible ones below $c_L \geq 16$ are exhausted by

c_L	$n = -2c_L$		
16	-32	$so(32)$,	$[so(16) \times so(16)]^\circ$
$\frac{31}{2}$	-31	$[(E_8)_2]^\circ$	
15	-30	$[su(16)]^\circ$	
14	-28	$[E_7 \times E_7]^\circ$	
12	-24	$[so(24)]^\circ$	
8	-16	E_8	
$\frac{1}{2}$	-1	ψ	

Here, $[\mathfrak{g}_k]^\circ$ is a fermionic modular-invariant extension of the current algebra $\mathfrak{g}_k$, where $k = 1$ is omitted for brevity.

The red ones have zero ordinary and/or mod-2 elliptic genus.

Let's compare with the Table in [Bruner-Rognes 2021]:



Note the perfect match with

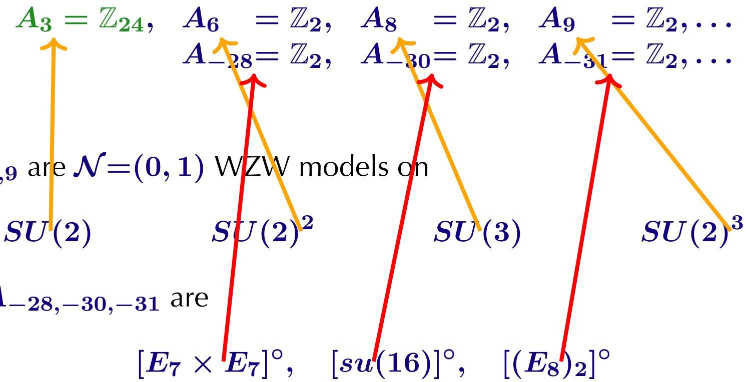
c_L	$n = -2c_L$	theory
$\frac{31}{2}$	-31	$[(E_8)_2]^\circ$
15	-30	$[su(16)]^\circ$
14	-28	$[E_7 \times E_7]^\circ$

!

I was totally shocked when I first noticed it while browsing the book.
They are very likely SQFT representatives of $A_{-28,-30,-31}$.

Let $\boldsymbol{A_d}$ be the subgroup of $\mathbf{TMF}_{\boldsymbol{d}}$
whose ordinary/mod-2 elliptic genus is zero.

In the range $-31 \leq d \leq 9$, the nonzero cases are:



$$\begin{aligned} &A_3 = \mathbb{Z}_{24}, \quad A_6 &&= \mathbb{Z}_2, \quad A_8 &&= \mathbb{Z}_2, \quad A_9 &&= \mathbb{Z}_2, \ldots \\ & \qquad\qquad\quad A_{-28} &&= \mathbb{Z}_2, \quad A_{-30} &&= \mathbb{Z}_2, \quad A_{-31} &&= \mathbb{Z}_2, \ldots \end{aligned}$$

$A_{3,6,8,9}$ are $\mathcal{N}{=}(0,1)$ WZW models on

$$SU(2) \qquad SU(2)^2 \qquad SU(3) \qquad SU(2)^3$$

and $A_{-28,-30,-31}$ are

$$[E_7 \times E_7]^\circ, \quad [su(16)]^\circ, \quad [(E_8)_2]^\circ$$

[YT-Yamashita 2305.06196]

A TMF pairing and the Green-Schwarz coupling

We were talking about the subgroup A_n of $\mathbf{TMF}_n$ for which ordinary and mod-2 Witten index is zero.

Mathematicians say that

$$A_d \longleftrightarrow A_{-22-d}$$

are Pontryagin dual if $d \not\equiv 3$ mod 24:

$$\begin{array}{llll} A_3 = \mathbb{Z}_{24}, & A_6 = \mathbb{Z}_2, & A_8 = \mathbb{Z}_2, & A_9 = \mathbb{Z}_2, \dots \\ & \updownarrow & \updownarrow & \updownarrow \\ & A_{-28} = \mathbb{Z}_2, & A_{-30} = \mathbb{Z}_2, & A_{-31} = \mathbb{Z}_2, \dots \end{array}$$

So there should be a pairing

$$\begin{array}{cccc}
 & A_6 = & A_8 = & A_9 = \\
\mathcal{N}=(0,1) & & & \\
\text{WZW model on} & SU(2)^2 & SU(3) & SU(2)^3 \\
 & \updownarrow & \updownarrow & \updownarrow \\
\text{purely left-moving modular-} & & & \\
\text{invariant fermionic CFT} & [E_7 \times E_7]^\circ, & [su(16)]^\circ, & [(E_8)_2]^\circ \\
 & = A_{-28} & = A_{-30} & = A_{-31}
\end{array}$$

What would this be, physically ?

The key to the question is that, **these spin-CFTs provide the angular part of the non-supersymmetric heteortic $p = 4$-, 6- and 7-branes of** [Kaidi-Ohmori-YT-Tachikawa 2303.17623].

$$\begin{array}{ccccc}
 & \underbrace{\mathbb{R}^{p,1}\times\mathbb{R}_{>0}} & \times & \underbrace{S^{8-p}+\text{current algebra}} & \\
 & & & \downarrow \text{RG} & \\
A_9 & d=9 & \leftrightarrow & [(E_8)_2]^\circ & A_{-31} \\
A_8 & d=8 & \leftrightarrow & [su(16)]^\circ & A_{-30} \\
A_6 & d=6 & \leftrightarrow & [E_7\times E_7]^\circ & A_{-28}
\end{array}$$

This arises **exactly on the places where** the pairing $A_d \leftrightarrow A_{-d-22}$ mathematicians constructed arises.

Concretely, take the pair

$$A_6 \quad d = 6 \quad \leftrightarrow \quad E_7 \times E_7 \quad A_{-28}$$

Question:

What would $A_6 \simeq \mathbb{Z}_2$ generated by

$$\mathcal{N}{=}(0,1) \text{ WZW model on } SU(2) \times SU(2)$$

provide for heterotic string compactification with $[E_7 \times E_7]^\circ$?

Answer:

There is a **discrete Green-Schwarz coupling**, which gives the phase -1, on the 6-dimensional manifold $SU(2) \times SU(2)$ with unit H flux on this heterotic string compactification with $[E_7 \times E_7]^\circ$.

In general, for a d-dimensional spacetime,
the internal CFT should have

$$c_L = 26 - d, \qquad c_R = \frac{3}{2}(10 - d)$$

therefore it is an element in

$$\mathrm{TMF}_{2(c_R - c_L) = -22-d}\,.$$

So, the discrete part of the Green-Schwarz coupling is a pairing

$$d \longleftrightarrow -22 - d.$$

This pair of dimensions agrees with what appears in the mathematical pairing :

$$A_d \longleftrightarrow A_{-22-d}.$$

So the natural guess is that the mathmatical pairing

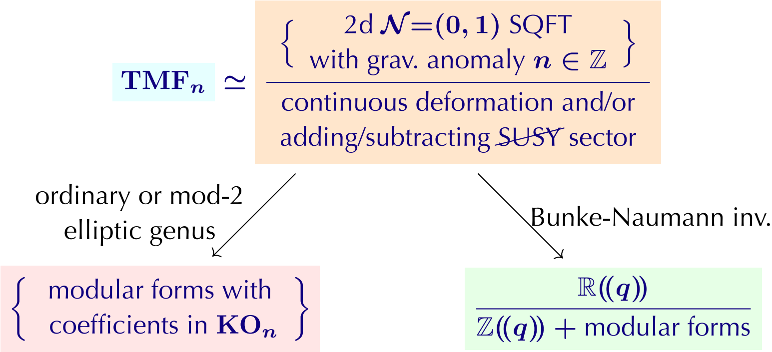
$$A_d \longleftrightarrow A_{-22-d}$$

is actually the discrete part of the Green-Schwarz coupling.

Together with Yamashita, I confirmed it in [YT-Yamashita 2305.06196].
It is written as a math paper with a short summary for physicists.

If you're interested, please have a look!

Summary



$$\mathbf{TMF}_{n} \simeq \frac{\left\{\begin{array}{c}\text{2d } \mathcal{N}=(\mathbf{0},\mathbf{1}) \text{ SQFT}\\ \text{with grav. anomaly } n\in\mathbb{Z}\end{array}\right\}}{\begin{array}{c}\text{continuous deformation and/or}\\ \text{adding/subtracting } \cancel{\text{SUSY}} \text{ sector}\end{array}}$$

ordinary or mod-2
elliptic genus

Bunke-Naumann inv.

$$\left\{\begin{array}{c}\text{modular forms with}\\ \text{coefficients in } \mathbf{KO}_{n}\end{array}\right\} \qquad \frac{\mathbb{R}((q))}{\mathbb{Z}((q)) + \text{modular forms}}$$

($n = \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{4}$ mod $\mathbf{8}$)

($n = \mathbf{3}, \mathbf{7}$ mod $\mathbf{8}$)

Let A_n be the kernel of $\mathrm{TMF}_n \to$ {ordinary or mod-2 elliptic genus}.
Nonzero A_n in the range $-31 \leq n \leq 9$ are:

$$\begin{tikzcd}
SU(2) \arrow[d] & SU(2)^2 \arrow[d] & SU(3) \arrow[d] & SU(2)^3 \arrow[d] & \\
A_3 = \mathbb{Z}_{24}, & A_6 = \mathbb{Z}_2, \arrow[d, leftrightarrow] & A_8 = \mathbb{Z}_2, \arrow[d, leftrightarrow] & A_9 = \mathbb{Z}_2, \arrow[d, leftrightarrow] & \dots \\
 & A_{-28} = \mathbb{Z}_2, & A_{-30} = \mathbb{Z}_2, & A_{-31} = \mathbb{Z}_2, & \dots \\
 & {[E_7 \times E_7]^\circ} \arrow[u] & {[su(16)]^\circ} \arrow[u] & {[(E_8)_2]^\circ} \arrow[u] &
\end{tikzcd}$$

$A_3 = \mathbb{Z}_{24}$ is detected by Bunke-Naumann invariant,
and the rest has the pairing

$$A_d \longleftrightarrow A_{-22-d}$$

which captures the discrete part of the Green-Schwarz coupling.