On fractional M5 branes and frozen singularities

Yuji Tachikawa

Sep. 2015, KIAS

cf. global models were already analyzed implicitly in

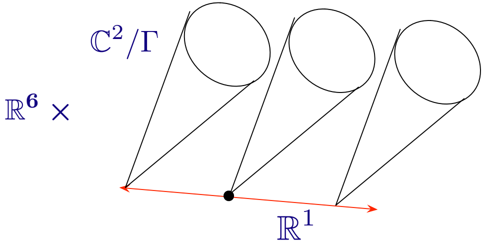
[Aspinwall-Morrison,
"Point-like instantons on K3 orbifolds",
hep-th/9705104]

[de Boer-Dijkgraaf-Hori-Keurentjes-Morgan-Morrison-Sethi,
"Triples, fluxes, and strings",
hep-th/0103170]

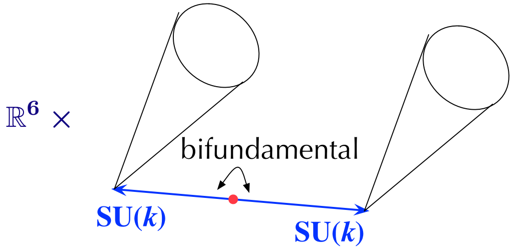
What I'm going to talk is a local analysis.

I′ve been studying 6d $\mathcal{N}=(1,0)$ theories for two years.

A large class of such theories can be obtained
by putting M5-branes on the ALE singularities:

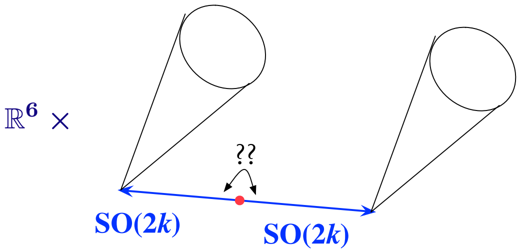


When $\Gamma = \mathbb{Z}_k$, we have $\mathbf{SU}(k)$ gauge fields at the singularity, and an M5 just gives a bifundamental of $\mathbf{SU}(k) \times \mathbf{SU}(k)$:

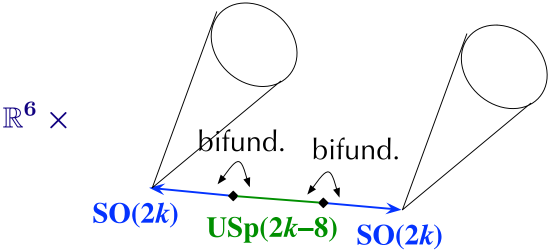


But surprising things happen when $\mathbf{\Gamma}$ is of type $\boldsymbol{D_k}$ or $\boldsymbol{E_k}$.
[del Zotto-Heckman-Tomasiello-Vafa, 1407.6359]

For example, take $\mathbf{\Gamma}$ of type $\boldsymbol{D_k}$ and put 1 M5:

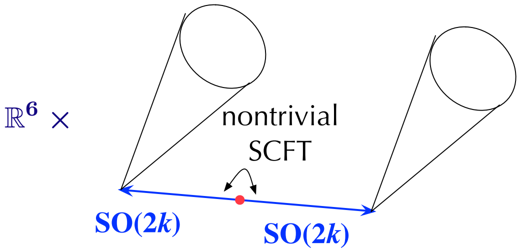


The M5 becomes two fractional M5s:

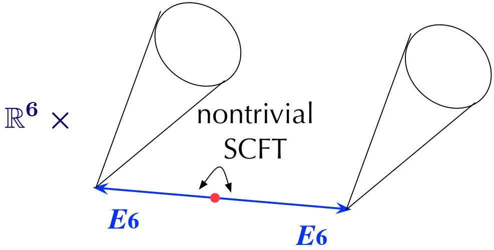


Somehow the middle region the gauge group is $\mathbf{USp(2}\boldsymbol{k}\mathbf{-8)}$,
and each half-M5 gives a bifundamental.

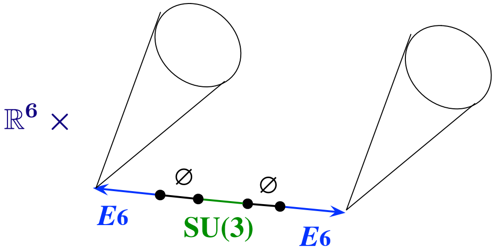
So if we merge two half-M5s back into one, we get a nontrivial SCFT:



Similarly, when $\mathbf{\Gamma}$ is of type $\boldsymbol{E_6}$, a full M5-brane fractionates ...



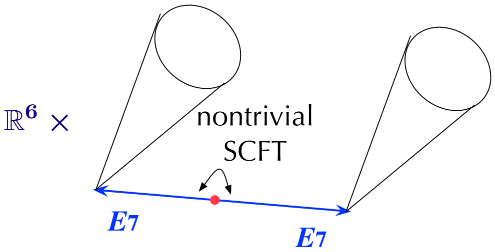
Similarly, when Γ is of type E_6, a full M5-brane fractionates ...



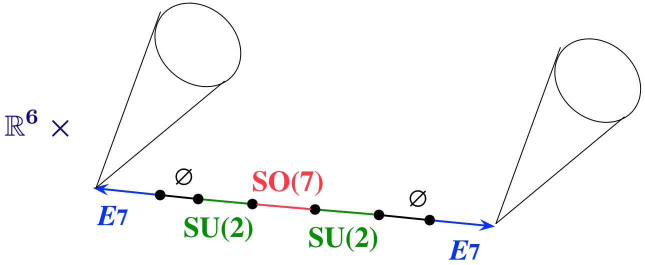
into 4 fractional M5s, and the gauge groups occur in the sequence

$$E_6, \quad \varnothing, \quad \mathbf{SU(3)}, \quad \varnothing, \quad E_6.$$

Similarly, when $\mathbf{\Gamma}$ is of type $\boldsymbol{E_7}$, a full M5-brane fractionates ...



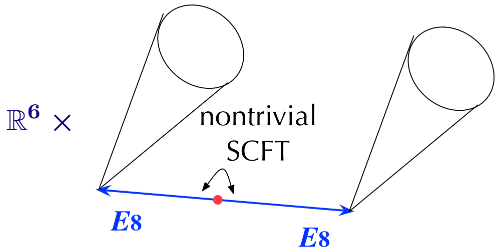
Similarly, when $\mathbf{\Gamma}$ is of type $\boldsymbol{E_7}$, a full M5-brane fractionates ...



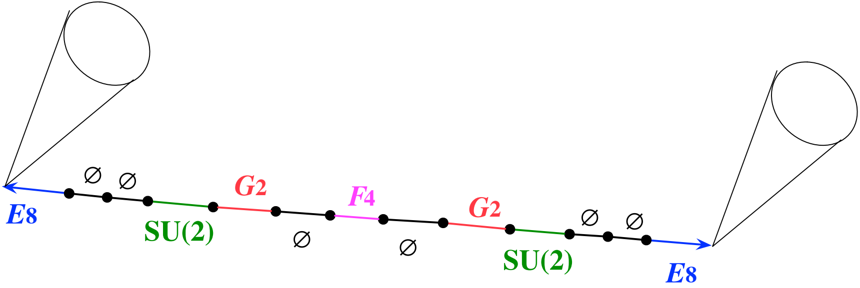
into 6 fractional M5s, and the gauge groups occur in the sequence

$$\boldsymbol{E_7}, \quad \varnothing, \quad \mathbf{SU(2)}, \quad \mathbf{SO(7)}, \quad \mathbf{SU(2)}, \quad \varnothing, \quad \boldsymbol{E_7}.$$

Finally, when Γ is of type E_8, a full M5-brane fractionates ...



Finally, when $\mathbf{\Gamma}$ is of type $\boldsymbol{E_8}$, a full M5-brane fractionates ...



into 12 fractional M5s, and the gauge groups occur in the sequence

$$\boldsymbol{E_8, \varnothing, \varnothing, \mathbf{SU(2)}, G_2, \varnothing, F_4, \varnothing, G_2, \mathbf{SU(2)}, \varnothing, \varnothing, E_8}$$

Summarizing, a full M5

- on type $\boldsymbol{A}$ singularities:
  doesn't fractionate.
- on type $\boldsymbol{D_k}$ singularities:
  fractionates into **2**.
  Groups: $\mathbf{SO(2}\boldsymbol{k}\mathbf{)}$, $\mathbf{USp(2}\boldsymbol{k}\mathbf{-8)}$, $\mathbf{SO(2}\boldsymbol{k}\mathbf{)}$
- on type $\boldsymbol{E_6}$ singularities:
  fractionates into **4**.
  Groups: $\boldsymbol{E_6}$, $\varnothing$, $\mathbf{SU(3)}$, $\varnothing$, $\boldsymbol{E_6}$.
- on type $\boldsymbol{E_7}$ singularities:
  fractionates into **6**,
  Groups: $\boldsymbol{E_7}$, $\varnothing$, $\mathbf{SU(2)}$, $\mathbf{SO(7)}$, $\mathbf{SU(2)}$, $\varnothing$, $\boldsymbol{E_7}$.
- on type $\boldsymbol{E_8}$ singularities:
  fractionates into **12**,
  Groups: $\boldsymbol{E_8}$, $\varnothing$, $\varnothing$, $\mathbf{SU(2)}$, $\boldsymbol{G_2}$, $\varnothing$, $\boldsymbol{F_4}$, $\varnothing$, $\boldsymbol{G_2}$, $\mathbf{SU(2)}$, $\varnothing$, $\varnothing$, $\boldsymbol{E_8}$.

What the heck is that?

Aim of the talk:

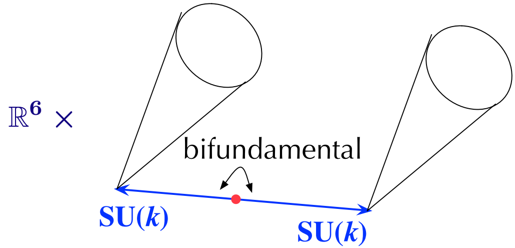
Better understand why this is the case.

Contents

Contents

For Γ of type A or D, one can just reduce the system to IIA.

For example, this becomes ...

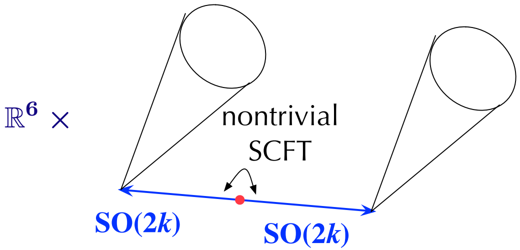


Just this:

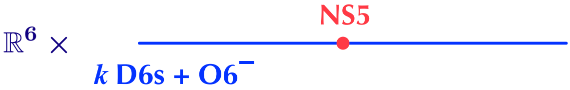
$\mathbb{R}^{6} \times$

which clearly doesn't fractionate.

When $\mathbf{\Gamma}$ is type $\boldsymbol{D}$, this becomes ...



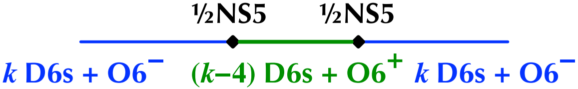
This:



which is known to fractionate to:

this:

$\mathbb{R}^6 \times$



Remember: O$p^{\pm}$ becomes O$p^{\mp}$ when we cross a half-NS5.

So far so good, but when Γ is of type $\boldsymbol{E}$, you can't reduce to IIA.

I say, type $\boldsymbol{A}$ and type $\boldsymbol{D}$ singularities are so **exceptional** that they don't show the generic behavior.

Type $\boldsymbol{E}$ is the generic case.

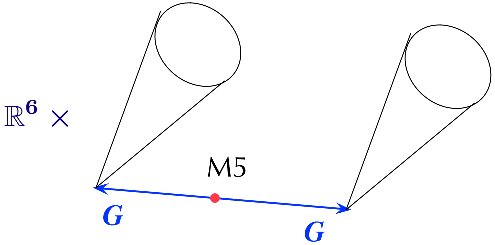
We can say we understand things **only when** we have a method equally applicable to all the types $\boldsymbol{A}$, $\boldsymbol{D}$, $\boldsymbol{E}$.

Contents

So, let's use F-theory.

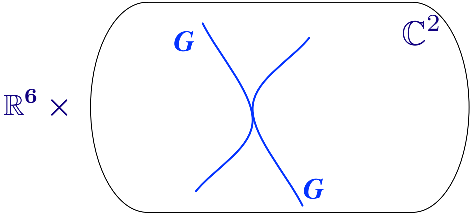
This is the method used by [del Zotto-Heckman-Tomasiello-Vafa, 1407.6359].

Recall that the M-theory configuration



is dual to ...

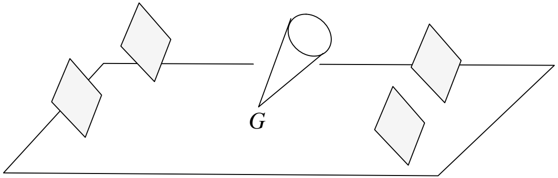
This F-theory configuration:



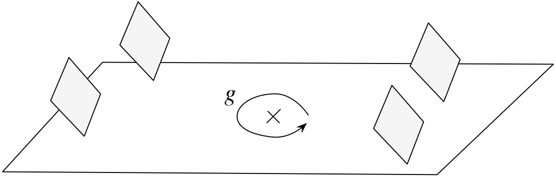
where two F-theory 7-branes intersect at a point.

Before getting further, let's recall the duality chain relating the two.

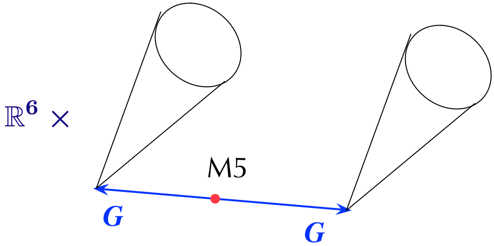
First, recall that a $\mathbb{C}^2/\Gamma$ singularity can be embedded in an elliptic fibration:



corresponding to a certain $\mathbf{SL(2,\mathbb{Z})}$ monodromy:



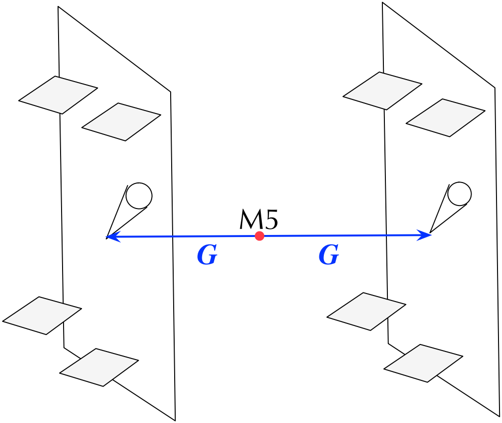
So the M-theory configuration



can be embedded into an elliptic fibration of the form ...

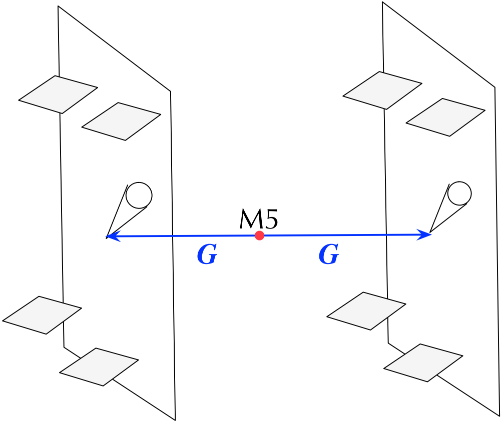
into an elliptic fibration of this form:

$\mathbb{R}^{\mathbf{6}} \times$



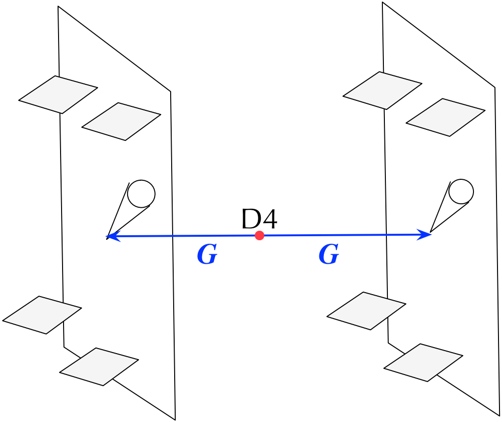
Compactify it on $\boldsymbol{S^1}$, which we make large at the last step again:

$\mathbb{R}^5 \times \boldsymbol{S^1} \times$

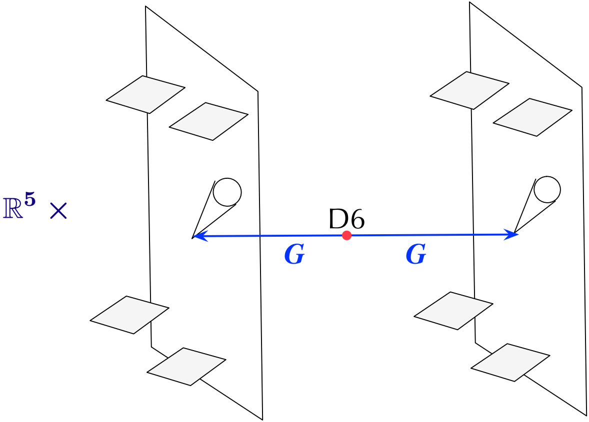


Reduce it to IIA:

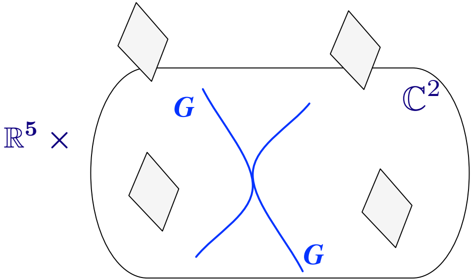
$\mathbb{R}^{5} \times$



Take the double T-dual along the elliptic fiber:



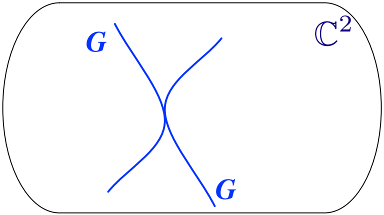
Lift it again to M-theory:



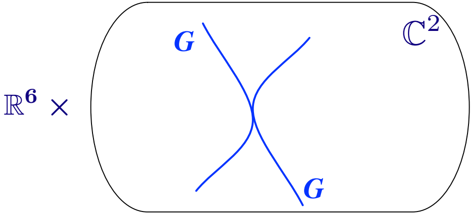
where $\mathbb{C}^2$ is better to be thought of as a Taub-NUT

This is the F-theory on

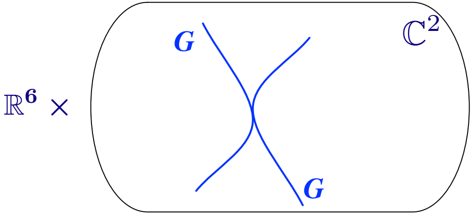
$$\mathbb{R}^5 \times S^1 \times$$



We make S^1 large again, done!



So, how do we know that something happens when $\boldsymbol{G}$ is not of type $\boldsymbol{A}$?



Recall that the elliptic fibration can be put to the Weierstrass form

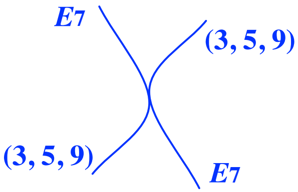
$$y^2 = x^3 + ax + b$$

where a, b are functions on the base.

Let $\Delta = 4a^3 + 27b^2$ be its discriminant.

	g	G	$\mathbf{ord}(a)$	$\mathbf{ord}(b)$	$\mathbf{ord}(\Delta)$
I_k	$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	$\mathbf{SU}(k)$	0	0	k
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$\varnothing$	1	1	2
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\mathbf{SU}(2)$	1	2	3
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\mathbf{SU}(3)$	2	2	4
I_k^*	$\begin{pmatrix} -1 & -k \\ 0 & -1 \end{pmatrix}$	$\mathbf{SO}(2k+8)$	2	3	$k+6$
IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	E_6	3	4	8
III^*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	E_7	3	5	9
II^*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	E_8	4	5	10

So, suppose two E_7 7-branes intersect.



Here $(3,5,9)$ means that (a, b, Δ) vanish to these orders there.

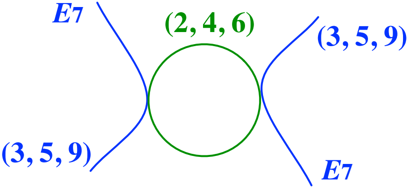
At the intersection,

$$(3,5,9) + (3,5,9) = (6,10,18) \geq (4,6,12).$$

A smooth elliptic fibration can't exceed $(4,6,12)$.

So we blow-up the intersection point.

We now get this configuration

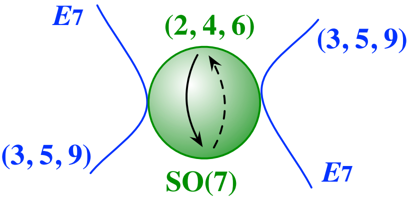


where

$$(2, 4, 6) = (3, 5, 9) + (3, 5, 9) - (4, 6, 12).$$

Looking up the table, this corresponds to I_0^* with $\mathbf{SO}(8)$.

A more detailed analysis using the Tate form (instead of the Weierstrass form) of the elliptic fibration shows that there is an outer-automorphism action of $\mathbf{SO(8)}$ around this S^2 of I_0^* curve



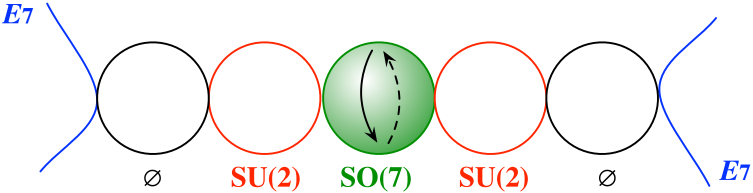
giving $\mathbf{SO(7)}$.

The intersection of $\mathbf{(2,4,6)}$ and $\mathbf{(3,5,9)}$ is still singular since

$$\mathbf{(2,4,6)+(3,5,9)\geq(4,6,12).}$$

We need to blow up, repeat ...

We end up with this final configuration:



So we can now work it out, for any $G = A_k$, D_k and $E_{6,7,8}$ in an uniform manner ...

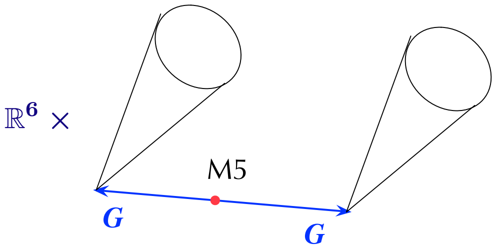
But I don't feel I understood it.

Let's try something else.

[Ohmori-Shimizu-YT-Yonekura, 1503.06217, Sec. 3.1]

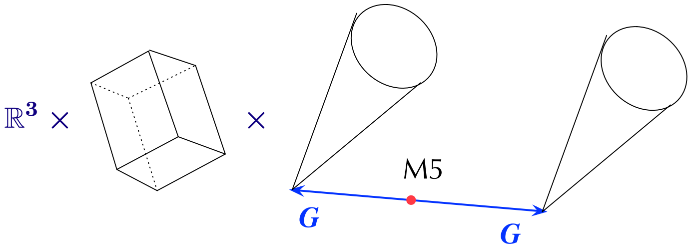
Contents

We start from the original setup:

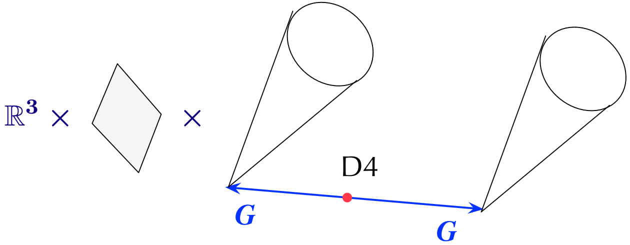


We're interested in the tensor branch of this 6d $\mathcal{N}=(1,0)$ theory.

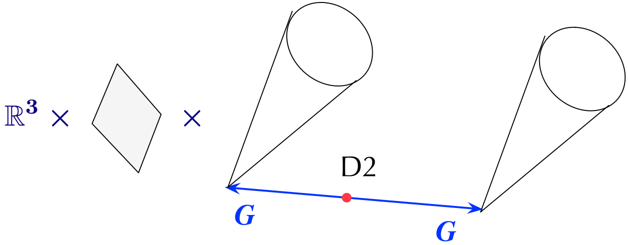
We can instead study the Coulomb branch of its $\boldsymbol{T^3}$ compactification:



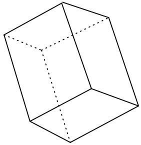
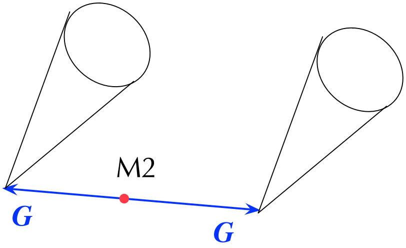
Reduce it to IIA:



Take the double T-dual:

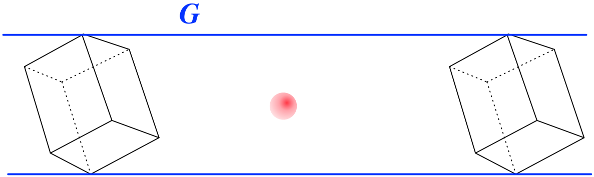


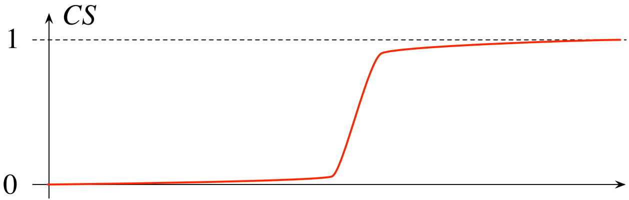
Lift it back to M-theory:

$\mathbb{R}^{\mathbf{3}} \times$  $\times$ 

We're now interested in its Higgs branch, since we've effectively taken the 3d mirror.

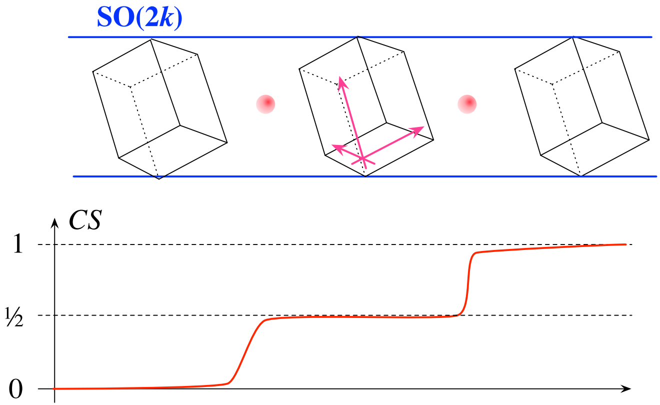
An M2 can dissolve into the G gauge field as an instanton on $T^3 \times \mathbb{R}$:



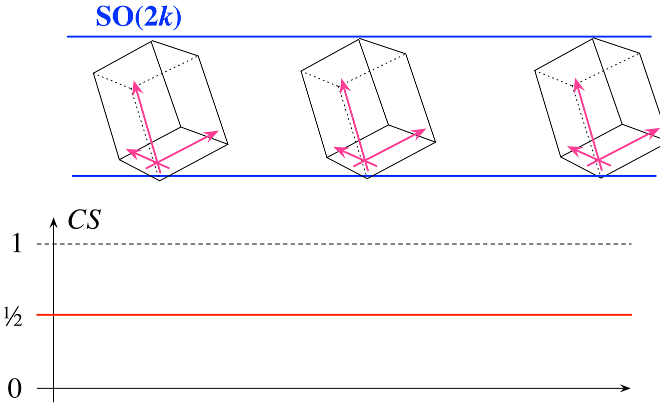


The plot below shows the evolution of the Chern-Simons invariant on T^3 at each slice.

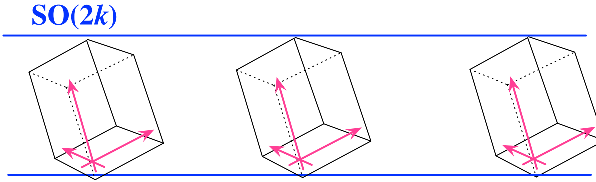
When $G = \mathbf{SO}(2k)$, the instanton can fractionate:



In an extreme situation, we have this:



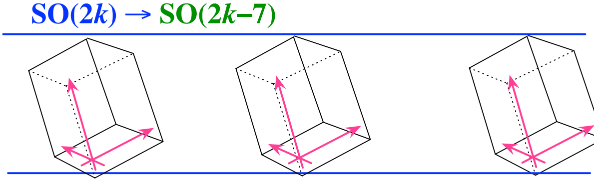
The bundle is flat but nontrivial.



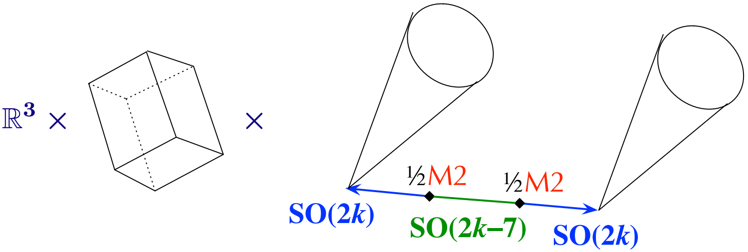
Three holonomies are known to be given by

$$\mathbf{diag}(+,+,+,-,-,-,-,+^{2k-7})$$
$$\mathbf{diag}(+,-,-,+,+,-,-,+^{2k-7})$$
$$\mathbf{diag}(-,+,-,+,-,+,-,+^{2k-7})$$

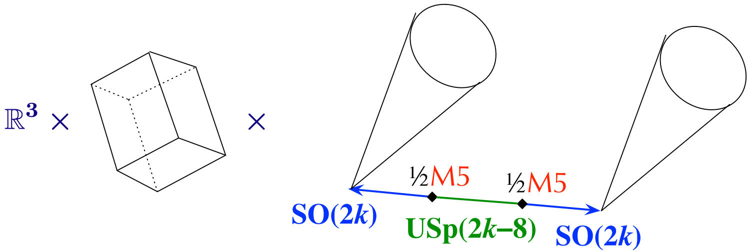
So the unbroken gauge group is



So we have

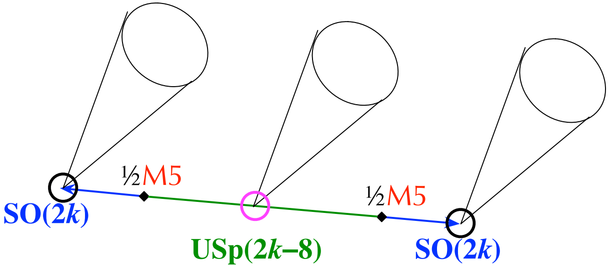


Going back the duality chain, we have



since we need to take 4d S-duality / 3d mirror symmetry:

$$\mathbf{SO(2}k\mathbf{-7)} \leftrightarrow \mathbf{USp(2}k\mathbf{-8)}$$

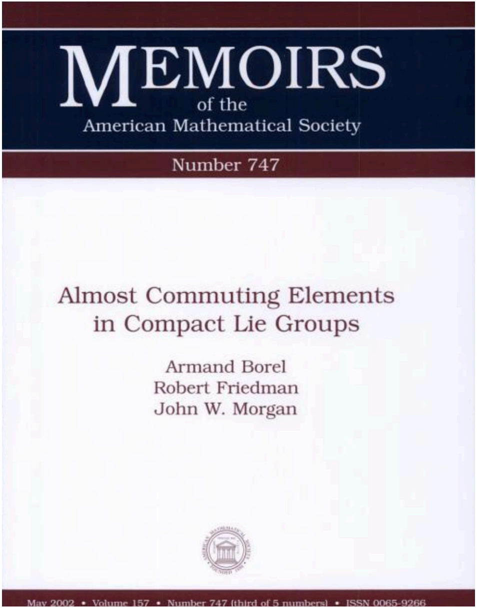


Note that

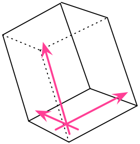
$$\int_{S^3/\Gamma} C = \begin{cases} 0 \mod 1 & \text{if } \mathbf{SO(2k)} \\ 1/2 \mod 1 & \text{if } \mathbf{USp(2k-8)} \end{cases}$$

In the latter case, the singularity is **partially frozen**.

The analysis can be carried out in a similar manner for any G, using the results in a monograph from 2002:



What needs to be done is the classification of flat G bundles on T^3

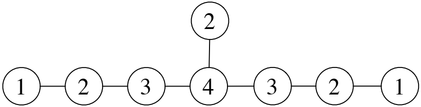


and the computation of their Chern-Simons invariants.

Summary of the facts:

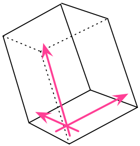
- $CS = n/d \mod 1$ where d appears as integer labels on the Dynkin diagram of type G and $\mathbf{gcd}(d, n) = 1$,
- The bundle is determined by d independent of G.

Example: $\boldsymbol{G = E_7}$. Allowed $\boldsymbol{d = 1, 2, 3, 4}$ since the Dynkin diagram is



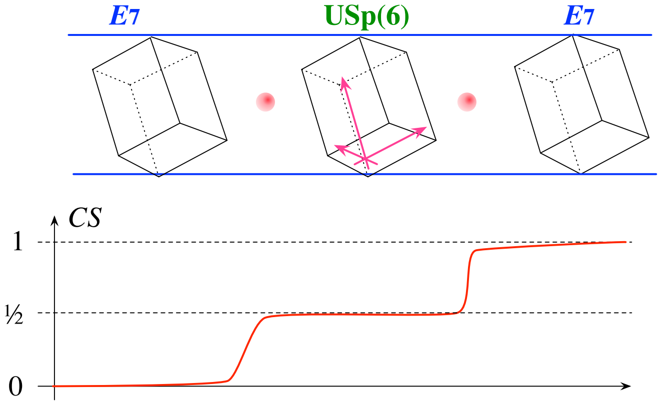
The bundle with $\boldsymbol{CS = 1/2}$ is still

$$\mathbf{diag}(+,+,+,-,-,-,-)$$
$$\mathbf{diag}(+,-,-,+,+,-,-)$$
$$\mathbf{diag}(-,+,-,+,-,+,-)$$

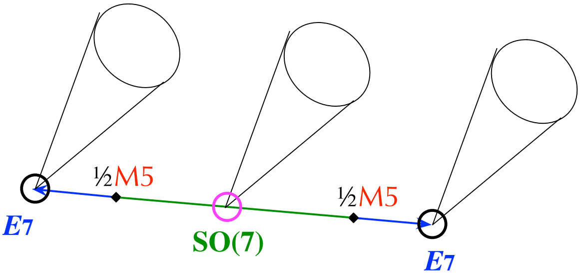


in $\mathbf{SO(7)}$. In fact they are in $\boldsymbol{G_2}$.

E_7 has a maximal subgroup $G_2 \times \mathbf{USp(6)}$. Therefore



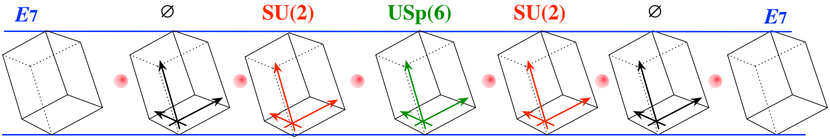
Taking the S-dual, we get

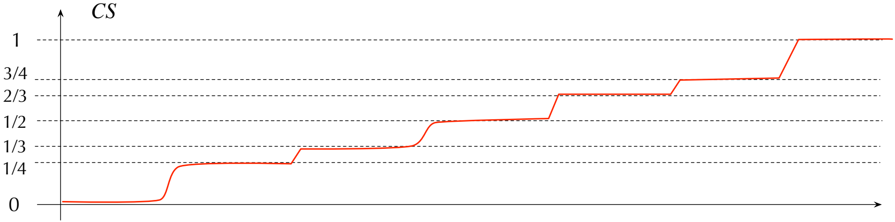


You can fractionate further, since allowed CS invariants are

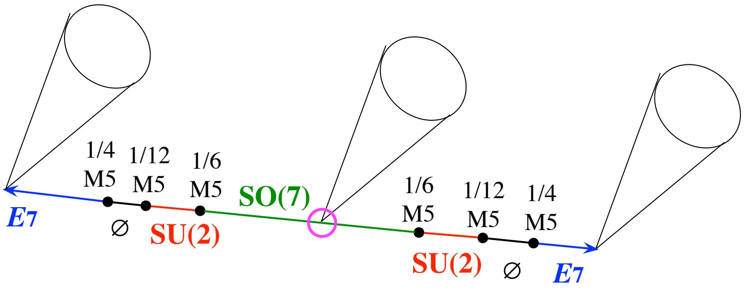
$$CS = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}.$$

We have

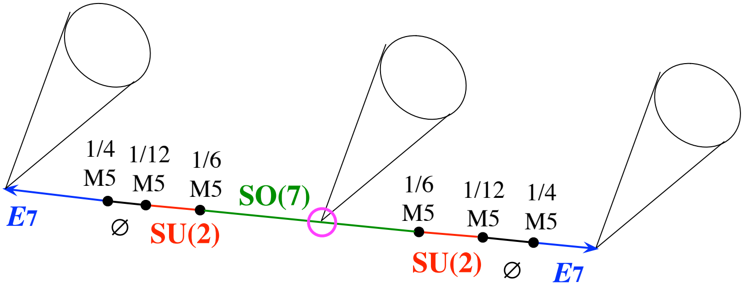




In the original duality frame we have



Note that the M5 charges are **not equally distributed**.

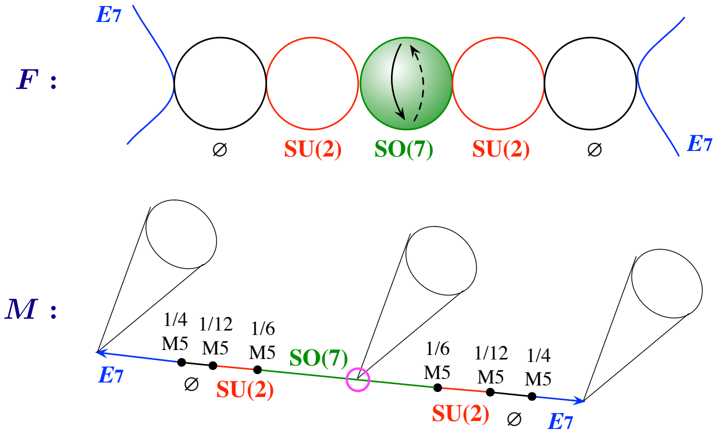


The rule is

$$\int_{S^3/\Gamma} C = \begin{cases} \mathbf{0} & \text{if } \boldsymbol{E_7} \\ \mathbf{1/2} & \text{if } \mathbf{SO(7)} \\ \mathbf{1/3, 2/3} & \text{if } \mathbf{SU(2)} \\ \mathbf{1/4, 3/4} & \text{if } \varnothing \end{cases}$$

Contents

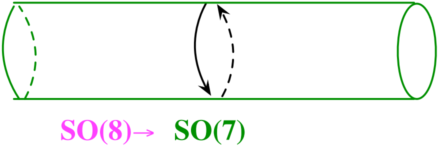
So we now have two ways to understand fractional M5s on ALE singularities:



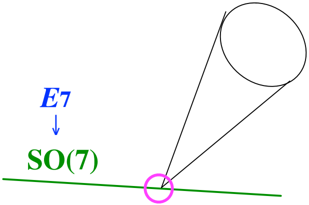
How are they related? [YT,1508.06679] [email discussions with A. Tomasiello]

We just have to show the equivalence of

$\boldsymbol{F}:\ \mathbb{R}^{6}\times$

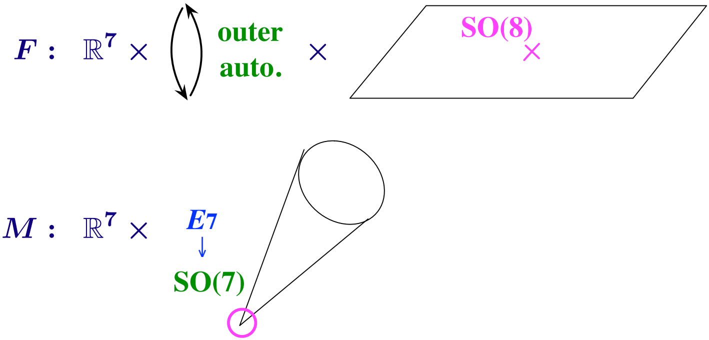


$\boldsymbol{M}:\ \mathbb{R}^{6}\times$



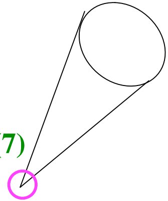
The rest is just a fiber-wise application of this duality.

We just have to show the equivalence of



Let's start from the M-theory side:

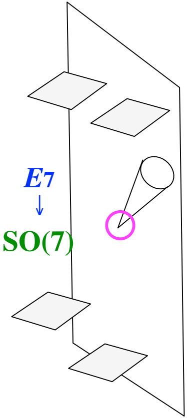
$\mathbb{R}^{7} \times$ E_7 $\downarrow$ $\mathbf{SO(7)}$



with $$\int_{S^3/\Gamma} C = 1/2.$$

Embed it in an elliptic fibration:

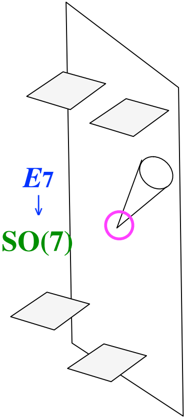
$\mathbb{R}^7 \times$



with $\int_{S^3/\Gamma} C = 1/2.$

Compactify it on S^1, which we enlarge again at the last step:

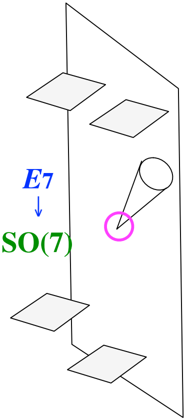
$\mathbb{R}^6 \times S^1 \times$



with $\int_{S^3/\Gamma} C = 1/2.$

Reduce it to IIA:

$\mathbb{R}^6 \times$

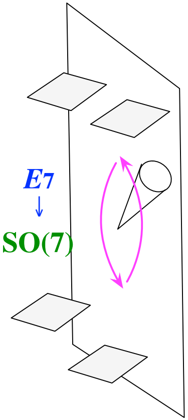


with $$\int_{S^3/\Gamma} C_{(3)} = 1/2$$

where $C_{(3)}$ is now the RR 3-form potential

Take the double T-dual:

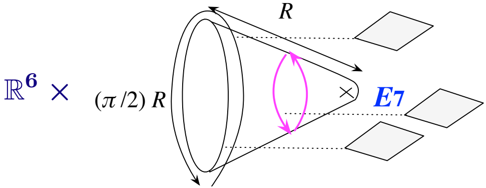
$\mathbb{R}^{6} \times$



with $\int_{S^1} C_{(1)} = 1/2$

where $C_{(1)}$ is now the RR **1**-form potential.

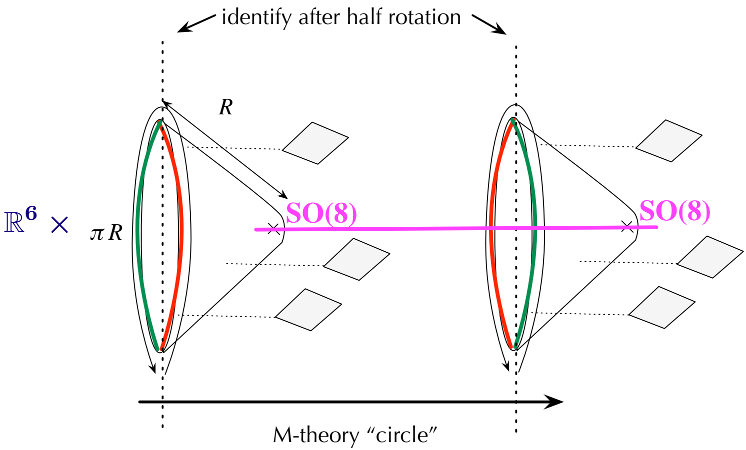
Recall that E_7 singularity is metrically a cone with opening angle $\pi/2$



with

$$\int_{S^1} C_{(1)} = 1/2.$$

When lifted to M-theory, this becomes



The opening angle should be $\boldsymbol{\pi}$, so the singularity should be $\mathbf{SO(8)}$.
The half-rotation involves $\mathbb{Z}_2$ outer-auto. of $\mathbf{SO(8)}$, thus giving $\mathbf{SO(7)}$.

$\mathbf{SL(2,\mathbb{Z})}$ monodromies also match:

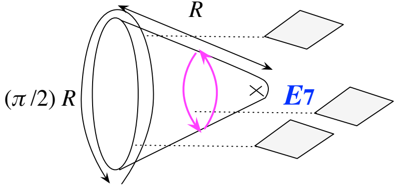
$$g_{E_7} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$g_{\mathbf{SO(8)}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we have

$$g_{\mathbf{SO(8)}} = {g_{E_7}}^2.$$

So the lift to M-theory of

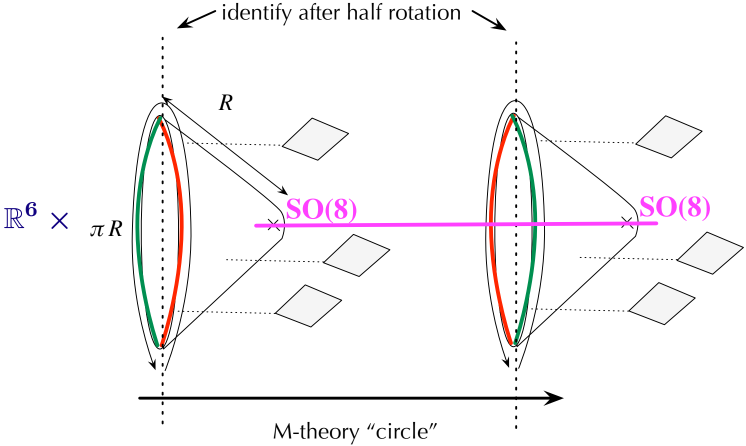
$\mathbb{R}^6 \times$



with

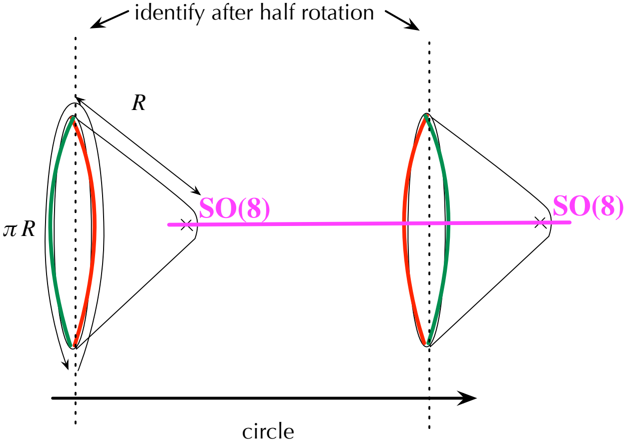
$$\int_{S^1} C_{(1)} = 1/2.$$

is M-theory on

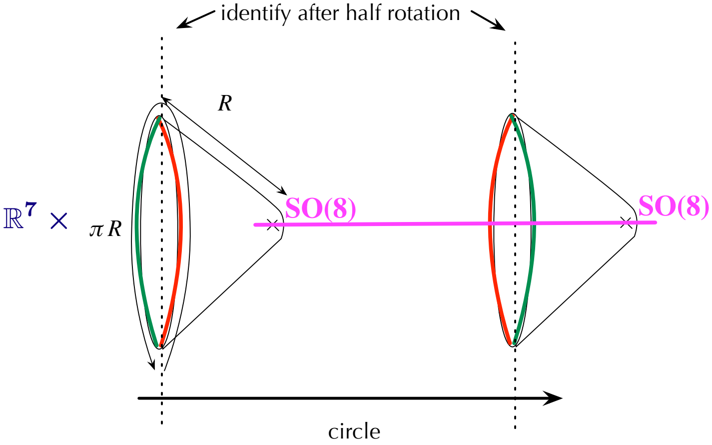


that is F-theory on

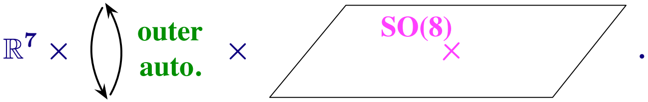
$\mathbb{R}^6 \times S^1 \times$



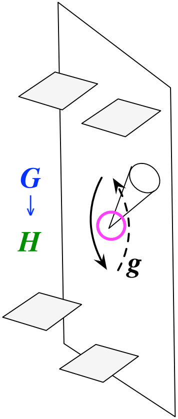
Making S^1 infinitely large, we have F-theory on



which was what we want to have:



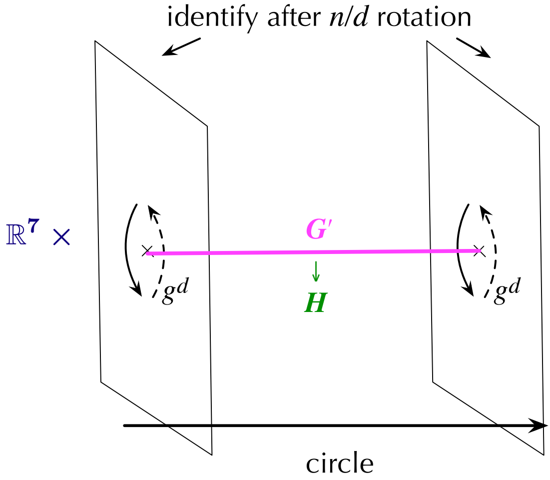
In general, M-theory on

$\mathbb{R}^7 \times$ 

with $\int_{S^3/\Gamma} C = n/d$

is dual to

F-theory on



So, given G and $r = n/d$,
there are two different ways to determine the gauge group H:

In M-theory, the steps are:

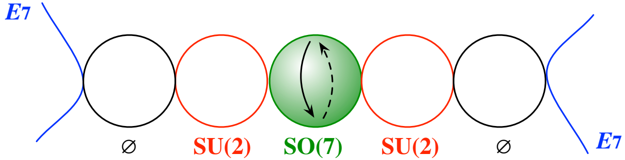
- Take the flat G-bundle P on T^3 with $CS = r$.
- Let G_P be the unbroken subgroup.
- Then H is the Langlands dual of G_P.

In F-theory, the steps are:

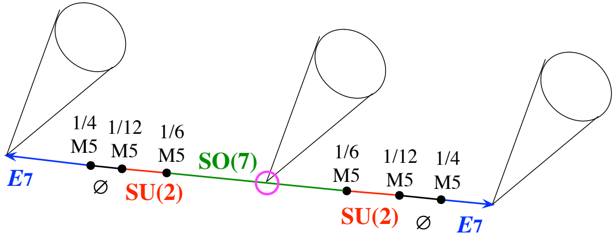
- Take the corresponding $\mathbf{SL(2,\mathbb{Z})}$ monodromy g.
- Let $g' = g^d$, and take the corresponding group G'.
- Take the invariant part H of G' under the outer-automorphism $\mathbb{Z}_d$.

They always agree!

Performing this duality fiber-wise, we have established the relation between F-theory on



and M-theory on



That's all what I wanted to say today.