Twenty years of the Seiberg-Witten theory

Yuji Tachikawa

Nov. 2014, Kyoto

This was the announced title, but please allow me to change it.

There are **so many concepts** named after Prof. Witten,

and **one can give a 40-min talk on** the impact of
any one of these concepts to theoretical physics.

I'd like to talk about
what I've been thinking for about a month.

Confinement and the Witten index

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Four fundamental forces in nature :

- **Gravitational force.**
  Things go down, since Earth pulls them.
- **Electromagnetic force.**
  The source of most of interactions around us.
- **'Weak' force.**
  Mediates β decay. $\sim$50 decays of K^{40} per sec. per kg
- **'Strong' force**
  Binds quarks into protons and neutrons.

Today I discuss (toy models of) the strong force.

The strong force has an intrinsic mass scale $\sim \mathbf{1\ GeV}$.

The strong force is described by the **Yang-Mills field** with the action

$$S = \frac{1}{g^2} \int d^4x \, \mathbf{tr} \, F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu]$$

where $A_{0,1,2,3}$ is in the adjoint of the gauge group G.

In the real world, $G = \mathbf{SU(3)}$.

Classically, the coupling constant g is dimensionless:
The mass scale "1GeV" **does not appear in the action**.

C. N. Yang gave a seminar on this topic in Feb. 1954,
at the Institute for Advanced Study.
(It's the Institute where Prof. Witten is one of the Professors.)

Here is a quote from his Collected Works.

> Soon after my seminar began, ... Pauli asked,
> "What is the mass of this field B_μ?"
> I said we did not know.
> Then I resumed my presentation,
> but soon Pauli asked the same question again.

> I said something ...
> I still remember his repartee: “That is not sufficient excuse.”
> I was so taken aback that I decided,
> after a few moments’ hesitation, to sit down.
> There was general embarrassment.
> Finally Oppenheimer said, “We should let Frank proceed.”

I should say, it’s reassuring that a great physicist
was also grilled during a seminar by an even greater physicist …

After 60 years, we now know that
correct quantum mechanical treatment of

$$S = \frac{1}{g^2} \int d^4x \, \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu]$$

generates a mass scale.

This is called the **confinement** of the Yang-Mills theory.

I should say that **there is no mathematically rigorous proof yet**.

Providing such a proof is one of
the **Clay Millennium Mathematical Prize Questions**.

Prove it right now, and get **$1,000,000!**

At least, people have put quantum-mechanical

$$S = \frac{1}{g^2} \int d^4x \, \mathbf{tr} \, F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu]$$

on a supercomputer, and **it indeed confined**.

But is there a more intuitive picture of what is going on?

Geraldus ’t Hooft suggested the following in the late 70s:

It is the **condensation** of **monopoles**.

What does this mean?

To understand what it means, let us recall the Higgs effect.

Suppose we have a $\mathbf{U(1)}$ gauge field $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ and a scalar field ϕ of electric charge q. The Lagrangian is then

$$\int d^4x(\frac{1}{g^2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi D^\mu\phi)$$

where

$$D_\mu\phi = \partial_\mu\phi + qA_\mu\phi.$$

Suppose further that ϕ condenses: $\langle\phi\rangle = v \neq 0$.

Then the term $D_\mu\phi D^\mu\phi$ gives the term $q^2v^2A_\mu A^\mu$, which means that the quanta of A_μ have masses $\sim qgv$.

The $\mathbf{U(1)}$ transformation by $|\omega| = 1$ acts on ϕ by

$$\phi \to \omega^q \phi.$$

When $\langle \phi \rangle = v$, the unbroken transformation is

$$\omega^q = 1,$$

i.e. we still have a topological $\mathbb{Z}_q$ gauge field.

For example, in a superconductor, the Cooper pair has charge $\mathbf{2}$, so you have $\mathbb{Z}_2$ gauge field.

The **Higgs effect** is due to
the condensation of **electrically charged** objects.

What 't Hooft suggested was that the **confinement of the Yang-Mills** is due to the condensation of **magnetically charged** objects.

This picture was **nicely confirmed for softly-broken $\mathcal{N}=2$ supersymmetric Yang-Mills**,
in [Seiberg-Witten, "Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory", hep-th/9407087]

But I don't have time to talk about that today.

Prof. Witten gave another important contribution in the early 80s.
["Constraints on Supersymmetry Breaking", Nucl.Phys.B202(1982)253]

Consider the Yang-Mills field $F_{\mu\nu}$ coupled to
a fermion λ in the adjoint representation (usually called gaugino):

$$S = \frac{1}{g^2} \int d^4x (\mathbf{tr}\, F_{\mu\nu} F^{\mu\nu} + \mathbf{tr}\, \bar{\lambda} \gamma^{\mu} D_{\mu} \lambda), \quad D_{\mu}\lambda = \partial_{\mu}\lambda + [A_{\mu}, \lambda]$$

We expect it to confine.

Let's say the gauge group is $\mathbf{SU}(N)$, for definiteness.

$$S = \frac{1}{g^2} \int d^4x (\mathbf{tr}\, F_{\mu\nu} F^{\mu\nu} + \mathbf{tr}\, \bar{\lambda} \gamma^\mu D_\mu \lambda), \quad D_\mu \lambda = \partial_\mu \lambda + [A_\mu, \lambda]$$

Important features:

- It has $\mathcal{N}=1$ **supersymmetry exchanging A_μ and λ**.
- Classically, it has a **$\mathrm{U}(1)$ symmetry $\lambda \to \omega\lambda$ for $|\omega| = 1$**.
- **Quantum mechanically**, it's a symmetry **only when $\omega^{2N} = 1$**.

Let us first concentrate on the last property:

It has a $\mathbb{Z}_{2N}$ symmetry

$$\lambda \to \omega\lambda, \qquad \omega^{2N} = 1$$

Now, *suppose* the system confines.
There will be non-zero gaugino condensate

$$\langle\lambda\lambda\rangle = S \neq 0$$

in the vacuum state.

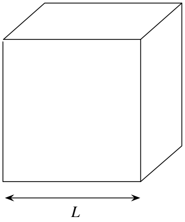
Applying the symmetry above, we see

$$\langle\lambda\lambda\rangle = S,\ \omega^2 S,\ \omega^4 S,\ \ldots,\ \omega^{2N-2} S$$

are equally possible.

So, there should be N **degenerate vacua, not just one.**

Put the system into a box of size L,
by imposing the periodic boundary condition under $x_i \to x_i + L$,
where L is very, very big.



There will be N zero energy states, since each of the vacuum with

$$\langle \lambda\lambda \rangle = \omega^{2k} S, \quad k = 0, \ldots, N-1$$

would give one zero energy state.

All this should be true *if* the system confines.

Is there a way to check the consistency of the ideas?

Prof. Witten had a brilliant idea:

- The number of the zero energy states, or more precisely the **index**, can be shown to be **independent of L**, using supersymmetry.
- Making L very, very small, we can directly compute the index, and show that it is indeed N.

When we put a supersymmetric system in a periodic box of size L, the system has the Hilbert space $\mathcal{H}$, on which we have three operators:

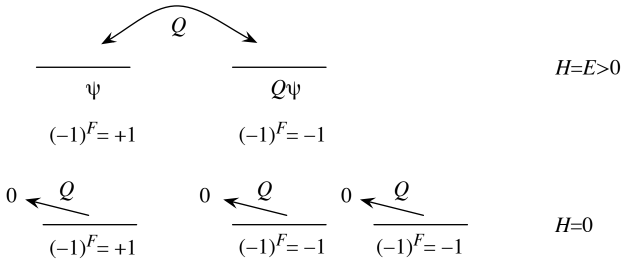
- $(-1)^F$, such that
  $(-1)^F = +1$ in a bosonic state,
  $(-1)^F = -1$ in a fermionic state.
- H, the Hamiltonian, whose eigenvalues are the energy of the states
- Q, the **supercharge**, such that

$$Q^2 = H, \qquad Q(-1)^F = -Q(-1)^F$$

Take a normalized eigenstate ψ of H with eigenvalue E:

$$E = \langle\psi|H|\psi\rangle = \langle\psi|QQ|\psi\rangle = \|Q|\psi\rangle\|^2 \geq 0$$

Therefore, we have:



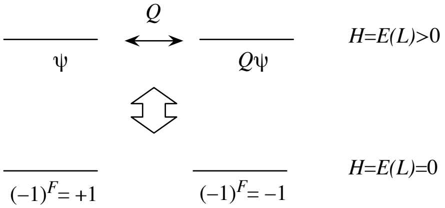
The Witten index Z is defined to be

the number of zero energy states with $(-1)^F = +1$
minus the number of zero energy states with $(-1)^F = -1$.

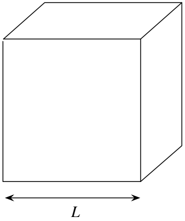
The Witten index,

the number of zero energy states with $(-1)^F = +1$
minus the number of zero energy states with $(-1)^F = -1$,

is **independent of the size L of the box**:



If the system confines, there are N vacua. In a very big box

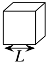


the index is

$$|Z(\text{big box})| = N.$$

The index is independent of N. So, ...

Even in a very small box,

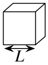


We should have

$$|Z(\text{small box})| = N.$$

Let's check this.

When the size L of the box is very small,



To have zero energy,
three holonomies $g_{1,2,3}$ of the gauge field around T^3 should commute.

For $G = \mathbf{SU}(N)$, they can be conjugated to

$$g_{1,2,3} \in T \subset \mathbf{SU}(N)$$

where T is the Cartan torus.

The fermionic zero modes λ_1, λ_2 take values in the Lie algebra of T.

The zero energy states have the wavefunction

$$|0\rangle, \quad (\lambda_1\lambda_2)|0\rangle, \quad (\lambda_1\lambda_2)^2|0\rangle, \ldots \quad (\lambda_1\lambda_2)^{N-1}|0\rangle$$

where $|0\rangle$ is a suitable wavefunction for the holonomies.

Note that $(\lambda_1\lambda_2)^N = 0$ because $\mathbf{rank}\, T = N - 1$.

Therefore, the index when L is very, very small is

$$|Z(\text{small box})| = 1 + \mathbf{rank}\, T = N.$$

This agrees with $|Z(\text{big box})|$ found before.

That was in **1982**.

In **1997**, Prof. Witten did the analysis for $G = \mathbf{Spin}(N)$, $N \geq 7$.
[“Toroidal compactification without vector structure”, hep-th/9712028]

If the system confines, the index when the box is large is

$$|Z(\text{large box})| = h^\vee(G) = N - 2.$$

How do we see this in the small box limit?

We need to study the commuting holonomies $g_{1,2,3} \in \mathbf{Spin}(N)$.

The moduli space $\mathcal{M}$ of three commuting holonomies g_1, g_2, g_3 is

$$\mathcal{M} = \mathcal{M}_0 \sqcup \mathcal{M}_1,$$

- $\mathcal{M}_0$ are those such that

$$g_i = t_i \in T \subset \mathbf{Spin}(N)$$

- $\mathcal{M}_1$ are those such that

$$g_i = \underline{g_i} t_i, \quad \underline{g_i} \in \mathbf{Spin}(7),\ t_i \in T' \subset \mathbf{Spin}(N-7).$$

Then the index when L is small is

$$\begin{aligned} |Z(\text{small box})| &= 1 + \mathbf{rank}\, T + 1 + \mathbf{rank}\, T' \\ &= (\lfloor \frac{N}{2} \rfloor + 1) + (\lfloor \frac{N-7}{2} \rfloor + 1) = N - 2. \end{aligned}$$

Nice.

Assuming that the super Yang-Mills confines for any G,
Prof. Witten then **derived a mathematical conjecture**

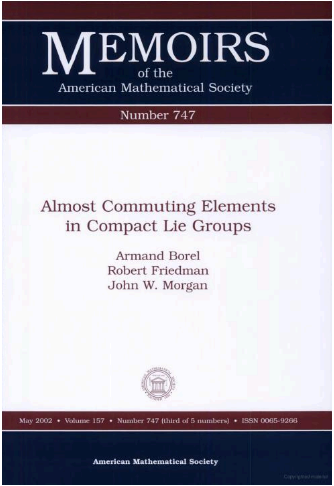
$$h^\vee(G) = \sum_a (\mathrm{rank}\, T_a + 1)$$

where the moduli space $\mathcal{M}$ of the commuting triple $g_{1,2,3} \in G$ has the form

$$\mathcal{M} = \sqcup_a \mathcal{M}_a$$

and T_a is the commutant of generic $(g_1, g_2, g_3) \in \mathcal{M}_a$.

This conjecture was proved in a monograph with $\sim$ **140** pages in **2002**



by Borel, Friedman and Morgan.

It's nice that

the experimental fact that the Yang-Mills theory confines

leads to

precise mathematical conjecture on the structure of groups

which is then

rigorously proved.

Isn't it?

Let me finally say a few words about what I was thinking last month.

What happens when $G = \mathbf{SU}(N)/\mathbb{Z}_N$ instead of $G = \mathbf{SU}(N)$?

In a small box, we still have the commuting triples (g_1, g_2, g_3)
of the form

$$g_i \in T \subset \mathbf{SU}(N)/\mathbb{Z}_N.$$

This still gives N zero energy states, just as before.

But there are more.

For simplicity just consider $G = \mathbf{SU(2)}/\mathbb{Z}_2 = \mathbf{SO(3)}$.

Then there is an isolated triple

$$\begin{aligned} g_1 &= \mathbf{diag}(+1, -1, -1) \\ g_2 &= \mathbf{diag}(-1, +1, -1) \\ g_3 &= \mathbf{diag}(-1, -1, +1) \end{aligned}$$

with the Stiefel-Whitney class on T^3 given by $w_2 = (-1, -1, -1)$.

In general, the Stiefel-Whitney class is $w_2 = (\pm 1, \pm 1, \pm 1)$ and there are **8** choices.

$w_2 = (+1, +1, +1)$ is the standard component with $T \subset \mathbf{SO(3)}$, giving **2** zero energy states.

7 other choices of w_2 give **1** zero energy state each.

In total, there are $2 + 7 = 9$ zero energy states in a small box.

So the Witten index should be **9** when the box is very, very large, too.

How can it be?

As we saw, there are two vacua with

$$\langle \lambda\lambda \rangle = +S, \quad -S$$

when $G = \mathbf{SU(2)}$. The same should be true for $G = \mathbf{SO(3)}$.

The point is that

the vacuum with $+S$ has a magnetic $\mathbb{Z}_2$ gauge symmetry, while the vacuum with $-S$ doesn't. [Aharony-Seiberg-YT, 2014]

Basically, the magnetic $\mathbf{U(1)}$ is broken by a monopole of charge 2 in the first vacuum.

Then, by choosing the holonomies of this $\mathbb{Z}_2$, we see there are

$$2^3 + 1 = 9$$

zero-energy states when the box is very, very big.

This is equal to $7 + 2$ we computed in the last slide.

This analysis can be generalized to arbitrary G/K
where G is simply-connected and K is a subgroup of the center of G.

In fact, the required computation was already done in Prof. Witten's
["Supersymmetric index in four-dimensional gauge theories",
hep-th/0006010]

So, technically, what I was thinking last month is **nothing new**.

The **existence of these discrete $\mathbb{Z}_k$ gauge freedom**
is very similar to what condensed matter theorists refer to as
symmetry protected topological phases these days.

Emphasizing a slightly different viewpoint
might not be completely useless.

Thanks for your attention.