Numerical evidence for a Haagerup CFT

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2110.03008

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(Phrases in purple are hyperlinked if you download the slides,
and see https://forvo.com/word/haagerup/ for the pronunciation.)

With four fantastic collaborators...

Tzu-Chen Huang

Ying-Hsuan Lin

Kantaro Ohmori

Masaki Tezuka

We obtained **numerical evidence** that there exists

a **1+1d conformal field theory** which has

the **Haagerup fusion category** as its **symmetry**.

We coordinated with another group to post two largely overlapping papers on the same day:

R. Vanhove, **L. Lootens**, **M. Van Damme**, **R. Wolf**, **T. J. Osborne**, **J. Haegeman**, **F. Verstraete**

A critical lattice model for a Haagerup conformal field theory

2110.03532

We were surprised that there was a competing group!

It also illustrates the fact that **any niche subject is liked by some people**, given the total number of human beings on the earth.

The Haagerup symmetry? Why do we care?

What is the model?

The Haagerup symmetry?
Why do we care?

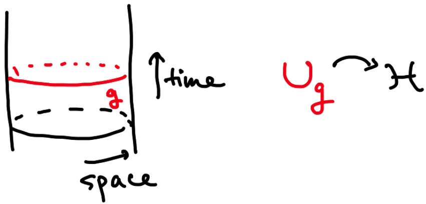
Symmetry is one of the fundamentals of physics.

Most often described by **groups**. But not always.

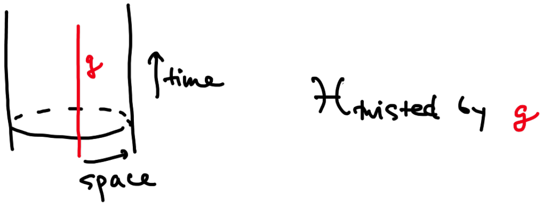
Various integrable spin chains are known to have **quantum group symmetry**, which is a deformation of **Lie group symmetries**.

The Haagerup symmetry is an example of **fusion category symmetries**, which generalize **finite group symmetries** in 1+1 dimensions.

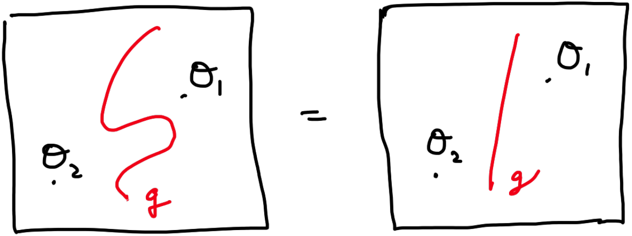
In 1+1d, a group action can be represented by a horizontal line:



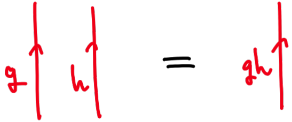
Twisted boundary conditions are represented by vertical lines:



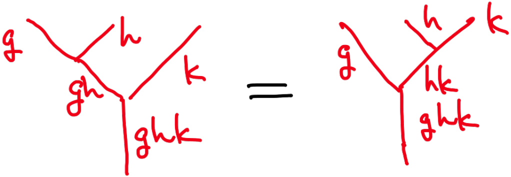
These lines are **topological**,
even when the 1+1d theory itself is non-topological:



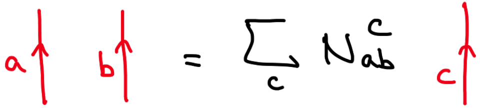
These topological lines can be fused:



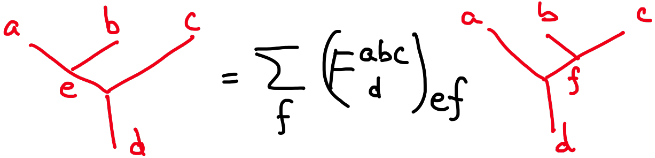
and this fusion is associative:



This can be generalized so that the fusion can be a sum



then the associativity is given by



which needs to satisfy the pentagon equation.

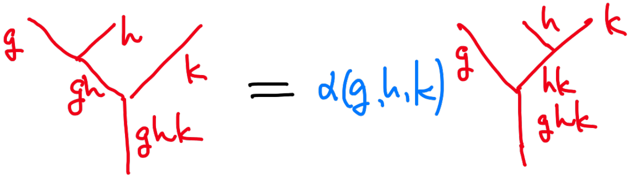
A package of topological walls with these data plus alpha gives a **generalized version of finite group symmetry** in 1+1d.

Mathematically given by a **unitary fusion category**.

Examples?

Finite groups G are examples, of course.

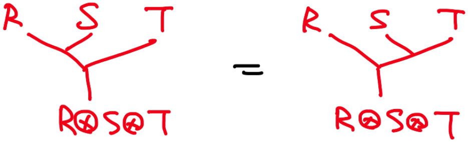
Finite groups G with nontrivial associativity rule:



$\alpha(g, h, k)$'s form a 3-cocycle, describing the symmetry G with anomaly $[\alpha] \in H^3(BG, U(1))$.

In a G-gauge theory, we can consider **Wilson lines** in the representation R of G.
They can be tensored

and the tensor product is associative



A theory $\boldsymbol{T}$ with $\boldsymbol{G}$ symmetry

$\downarrow$ gauge $\boldsymbol{G}$

A theory $\boldsymbol{T' = T/G}$ with "$\mathbf{Rep}(\boldsymbol{G})$ symmetry"

$\downarrow$ "gauge $\mathbf{Rep}(\boldsymbol{G})$"

Comes back to the original theory $\boldsymbol{T}$

T: a theory with a finite symmetry group G.

Pick a subgroup $H \subset G$, and gauge=orbifold it.

The gauged theory T/H has symmetry group

$$(\text{Normalizer of } H \text{ in } G)/H.$$

This process becomes, in terms of fusion categories...

T: a theory with symmetry fusion category C.

To gauge, pick an **algebra object** A in C.

The gauged theory T/A has symmetry given by fusion category

$$C' := C/A := \mathbf{Bimod}(A)$$

C' always has a 'dual' algebra object A' so that

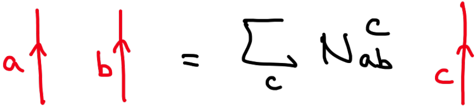
$$C = C'/A'.$$

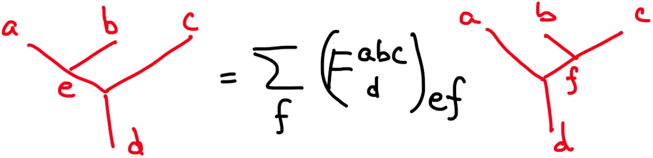
You can always 'ungauge' any gauging process.

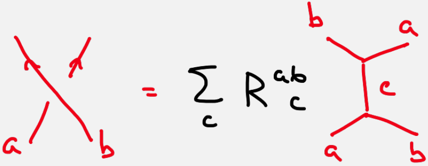
This physical interpretation of fusion-category-theoretic operations was found in [Carqueville-Runkel, 1210.6363].

Also recall Professor Wang's talk yesterday for the 2+1d case.

Anyons = fusion category + braiding







A package of anyon data is
called a **modular tensor category**.

Forgetting the braiding R^{ab}_c, we have a **fusion category**.

Representations of a rational **chiral conformal field theory** form a **modular tensor category**.

It is a big question whether all modular tensor categories arise in this way.

Anyway, taking affine $\mathfrak{g}$ algebra at level k,
we get a corresponding modular tensor category $\mathfrak{g}_k$.

This is also a fusion category.

Physically, G_k Wess-Zumino-Witten model has
$\mathfrak{g}_k$ as its fusion category symmetry.

This was originally noticed by [Verlinde (1988)].

So, there are two natural sources of fusion categories:

- **Groups with anomalies,**
- **Rational chiral conformal field theories.**

From any of these 'seed' fusion category C,
we can perform a generalized gauging
by picking an algebra object A.

This gives rise to a new fusion category $C' = C/A$.

In fact **almost all known concrete fusion categories are of this form**.

There are exceptions, of course.
The **Haagerup fusion category** is one of them.

The final source of fusion categories is **subfactors**,
a topic in operator algebras, introduced by [V. F. R. Jones (1983)].

A subfactor is a pair

$$N \subset M$$

of $\mathbf{II_1}$ factors, certain subalgebras of operators on Hilbert spaces.

Consider N-N bimodules coming from irreducible decompositions of

$$\underbrace{M \otimes_N M \otimes_N M \otimes_N \cdots \otimes_N M}_{\text{arbitrary number}}$$

These bimodules form a unitary fusion category C.

M viewed as an N-N bimodule is an algebra object $A = M$ in C.

So, a subfactor $N \subset M$ gives rise to the pair

(generalized symmetry C, generalized gauging A).

The property of such pairs was axiomatized by [Ocneanu (1988)], who called them **paragroups**.

Modular tensor categories also go back to [Moore-Seiberg (1988)].

The basic paper which treated fusion categories as such was [Etingof-Nikshych-Ostrik, math.QA/0203060]

Forgetting the algebra object / braiding took 14 years!!

Paragroups were described in detail in [Evans-Kawahigashi (1998)].

In fact, the correspondence

$$\text{subfactors} \longleftrightarrow \text{paragroups}$$

is one-to-one [Popa (1994)].

So, giving a subfactor $\boldsymbol{N} \subset \boldsymbol{M}$ is equivalent to
specifying the paragroup $(\boldsymbol{C}, \boldsymbol{A})$ where

- $\boldsymbol{C}$: fusion category = generalized symmetry
- $\boldsymbol{A}$: algebra object in $\boldsymbol{C}$ = generalized gauging of $\boldsymbol{C}$

The Jones index of the subfactor **is the quantum dimension of $\boldsymbol{A}$**.

It has been of interest to operator algebraists
to classify subfactors of small Jones index.

Subfactors of index ≤ 4 have been classified.
They can be described by Dynkin diagrams.

The next one is the Haagerup subfactor of index $\frac{5+\sqrt{13}}{2} \sim 4.3$.
[Haagerup (1993)] [Asaeda-Haagerup math.OA/9803044]

This corresponds to a pair $(C =: H_1, A)$.

The generalized gauging by A gives another fusion category $H_2 := H_1/A$.

There is one and only one additional algebra object A' in H_1, giving rise to $H_3 := H_1/A'$. [Grossman-Snyder 1102.2631]

Let us recap.

Fusion categories generalize **finite group** symmetries.

Most **fusion categories** come from
finite groups G or **affine Lie algebras** $\mathfrak{g}_k$.

There are a few others, such as the **Haagerup fusion categories** $H_{1,2,3}$.

Are there physical models
which have $H_{1,2,3}$ as fusion category symmetries?

The theory of subfactors was also the source of the celebrated **Jones polynomial**.

From a modern perspective,
knot polynomials come from anyons = modular tensor categories.

A subfactor $N \subset M$ gives the pair (C, A).

The so-called Drinfeld center construction gives
a **modular tensor category** $Z(C)$ from any **fusion category** C.

The center is independent of the generalized gauging,
i.e. $Z(C) = Z(C/A)$.

Are there physical models
which have $Z(H_1) = Z(H_2) = Z(H_3)$ as anyons?

So we have two questions:

Are there physical models which have $H_{1,2,3}$ as fusion category symmetries?

Are there physical models which have $Z(H_1) = Z(H_2) = Z(H_3)$ as anyons?

Topological theories always exist
for arbitrary fusion category (1+1d)

[Thorngren-Wang 1912.02817]
[Huang-Lin-Seifnashri 2110.02958]

and for arbitrary modular tensor category (2+1d)

[Witten (1989), Reshetikhin-Turaev (1991)],
[Turaev-Viro (1992), Ocneanu (1993)].

So, nontrivial questions I would like to pose are:

Are there 1+1d conformal field theories which have $H_{1,2,3}$ as fusion category symmetries?

Are there 1+1d rational conformal field theories which have $Z(H_1) = Z(H_2) = Z(H_3)$ as anyons?

Are there 1+1d conformal field theories which have $H_{1,2,3}$ as fusion category symmetries?

Are there 1+1d rational conformal field theories which have $Z(H_1) = Z(H_2) = Z(H_3)$ as anyons?

Historically **the latter question** has been better studied, e.g. [Evans-Gannon, 1006.1326]

Today we provide numerical evidence for **the first question**.

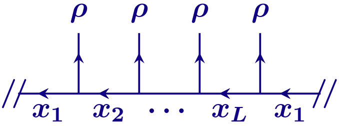
What is the model?

Are there **1+1d conformal field theories** which have $H_{1,2,3}$ as fusion category symmetries?

Our idea is to use the "anyon chain", introduced by [Feiguin, Trebst, Ludwig, Troyer, Kitaev, Wang, Freedman cond-mat/0612341]

This is a 1-dimensional spin chain whose basis states are of the form

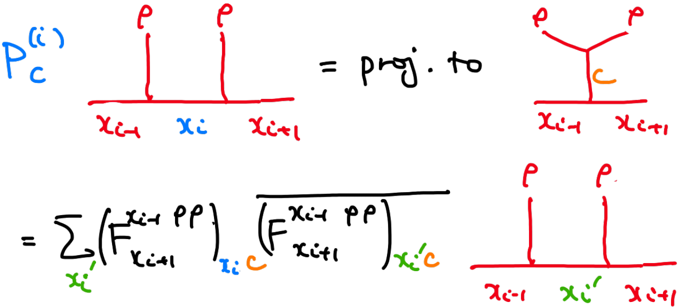
$$|x_1 x_2 \cdots x_L\rangle =$$



where ρ is a fixed anyon and x_i are anyon labels such that the fusion $x_i \rho$ contains x_{i+1}.

There are L sites.

A natural 'nearest-neighbor' interaction is given by the projection to the fusion channel c:



A natural class of Hamiltonians is then

$$H = \sum_c J_c \sum_i P_c^{(i)}$$

The original paper assumed that
the input data formed a **modular tensor category**.

It turns out that the braiding was not used, so **any fusion category will do**.

(So calling it an anyon chain is somewhat of a misnomer.
[Buican-Gromov 1701.02800] called it a fusion chain).

The resulting model has the input fusion category as its symmetry.

So, the spin chain Hamiltonian

$$H = \sum_c J_c \sum_i P_c^{(i)}$$

based on the Haagerup fusion category will do the job...

if it becomes a conformal field theory in the continuum limit!

Before continuing, let me comment on the competing work.

There is in general a correspondence between **1d quantum spin chains** and **2d statistical mechanical models**.

The 2d stat. mech. models for the 'anyon chain' was introduced independently by [Aasen-Fendley-Mong 2008.08598] and by [Lootens-Fuchs-Haegeman-Schweigert-Verstraete 2008.11187].

The paper by the competing group, [Vanhove-Lootens-Van Damme-Wolf-Osborne-Haegeman-Verstraete 2110.03532], studied the transfer matrix of this 2d model.

They are experts in this numerical field while **we are novices**, so their data are far cleaner.

We used numerical approaches to study this issue.

There are two steps involved:

1. Write down $\boldsymbol{H}$ explicitly.
2. Study $\boldsymbol{H}$ numerically.

Write down H explicitly

The Haagerup fusion category H_3 contains six simple objects,

$$1, a, a^2, \rho, a\rho, a^2\rho$$

with the fusion rule

$$a^3 = 1, \qquad a\rho = \rho a^2, \qquad \rho^2 = 1 + (1 + a + a^2)\rho.$$

We then need the associators = F-symbols.

Unfortunately operator algebraists hadn't written them down.
Recently done by three groups:

- [Titsworth (2018)] ← this was unpublished.
- [Osborne-Stiegemann-Wolf 1906.01322] ← by the other group.
- [Huang-Lin 2007.00670] ← by my collaborators.

Study H numerically

There are two standard methods:

- Exact diagonalization.
- Density-matrix renormalization group (DMRG).

Exact diagonalization

Up to a reasonable length L, we can simply diagonalize H numerically.

We went to $L = 18$, for which the dimension of the Hilbert space is

$$\sim (\frac{3+\sqrt{13}}{2})^{18} \sim 2 \cdot 10^9.$$

There are standard numerical algorithms to find **not all but the lowest N** eigenvalues and their eigenvectors.

We needed $N \sim 1000$.

Still this required us to use a computer
with 1.5TB of memory for about a week.

DMRG

Density-matrix renormalization group was introduced by [White (1992)].

A recent reformulation is that **it is a variational algorithm** looking for eigenvectors within the ansatz of **matrix product states**.

This can be used to find the ground state and a few excited states.

There is a very user-friendly package

ITensor

allowing you to do DMRG without understanding it.

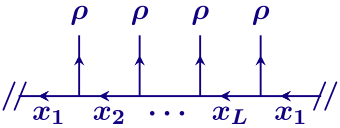
We went to $\boldsymbol{L = 36}$ in the periodic chain
and to $\boldsymbol{L = 144}$ in the open chain.

Numerical results

Recall that H_3 has simple objects $1, a, a^2, \rho, a\rho, a^2\rho$.

We used ρ to define the basis states:

$$|x_1 x_2 \cdots x_L\rangle =$$

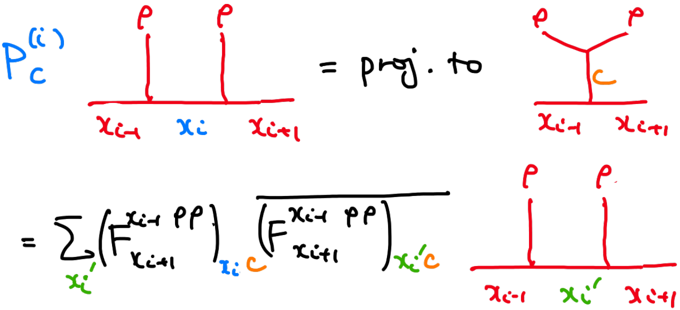


such that the fusion $x_i\rho$ contains x_{i+1}.

The Hamiltonian is

$$H = \sum_c J_c \sum_i P_c^{(i)}$$

where $P_c^{(i)}$ was



Of course we tried $H = \pm \sum_i P_1^{(i)}$ first. Didn't work.

Also reported in [Osborne, talk at Oberwolfach, 2019],
(organized by Kawahigashi-sensei)
and in [Wolf, 2101.04154] (who was a student of Osborne).

The next obvious choice was to try $H = -\sum_i P_\rho^{(i)}$.

It works, with central charge $c \sim 2$.

Let me provide a bit more detail.

Given a spin chain of length L, let $E_0(L)$ and $E_1(L)$ be the **energy of the ground state** and **that of the first excited state**. We would like to know the behavior in $L \to \infty$.

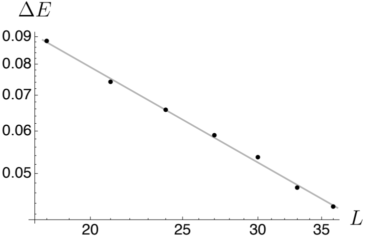
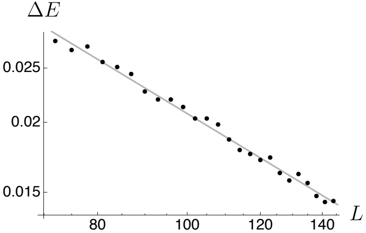
In a gapped phase,

$$E_1(L) - E_0(L) \to \text{const.}$$

while in a gapless phase

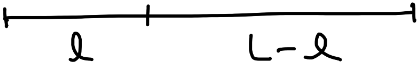
$$E_1(L) - E_0(L) \to L^{-z}$$

where $z = 1$ for ordinary CFT.

Periodic	Open
$\log \Delta E = -1.01(2) \log L + 0.49(8)$	$\log \Delta E = -1.0(2) \log L + 0.73(9)$
	

$E_1(L) - E_0(L) \sim L^{-1}$, as expected for a CFT.

In a CFT, the **entanglement entropy** of the ground state

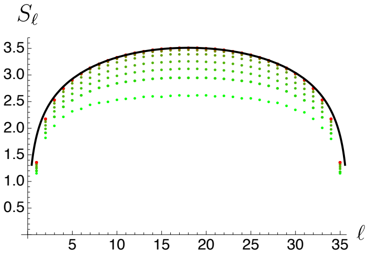


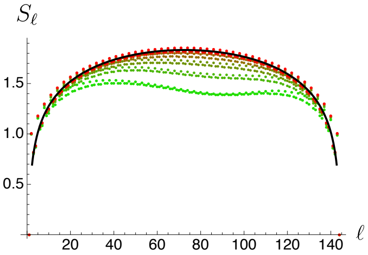
when you split the chain to segments of ℓ and $L - \ell$ is

$$S_\ell = \begin{cases} \frac{c}{3} \log\left(\frac{L}{\pi} \sin\frac{\pi\ell}{L}\right) + \mathrm{const}\,, \\ \frac{c}{6} \log\left(\frac{L}{\pi} \sin\frac{\pi\ell}{L}\right) + \mathrm{const}\,. \end{cases}$$

[Holzhey-Larsen-Wilczek, hep-th/9403108]
[Calabrese-Cardy, hep-th/0405152]

Periodic	Open
$c = 2.034(4)$	$c = 2.11(7)$





Changing colors show the DMRG sweeps searching for the ground state.

Using the state-operator correspondence,
an operator of dimension Δ_i appears as a state with

$$E_i(L) = \alpha L + \frac{v}{L}(\Delta_i - \frac{c}{12}) + o(\frac{1}{L}).$$

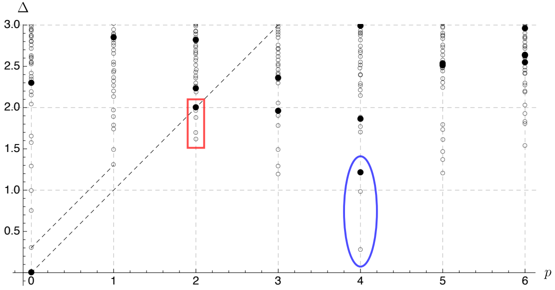
α and vc can be found by fitting $E_0(L)$, for which $\Delta_0 = 0$.

c was found from the entanglement entropy.

Δ_i can then be read off.

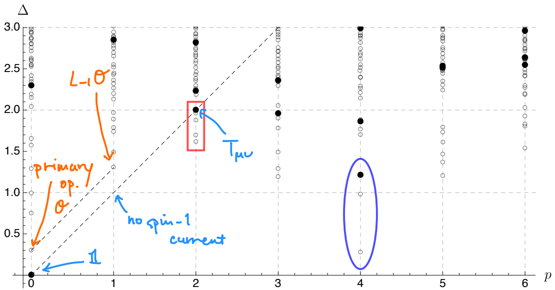
(Due to numerical issues we actually employed
a slightly different method to determine the parameters,
but you get the idea.)

$L = 12$



The ρ can take two values, $\rho_{\pm} = \dfrac{3 \pm \sqrt{13}}{2}$.
The filled/hollow dots are for ρ_{+}/ρ_{-}.

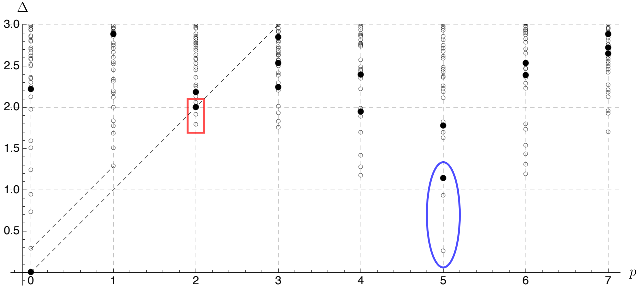
$L = 12$



The ρ can take two values, $\rho_\pm = \dfrac{3 \pm \sqrt{13}}{2}$.

The filled/hollow dots are for ρ_+/ρ_-.

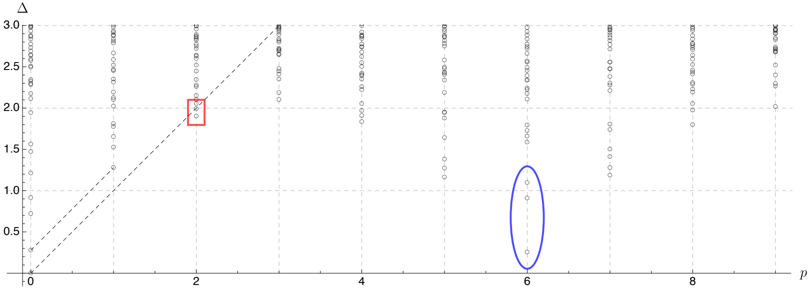
$L = 15$



The ρ can take two values, $\rho_{\pm} = \dfrac{3 \pm \sqrt{13}}{2}$.

The filled/hollow dots are for ρ_{+}/ρ_{-}.

$L = 18$

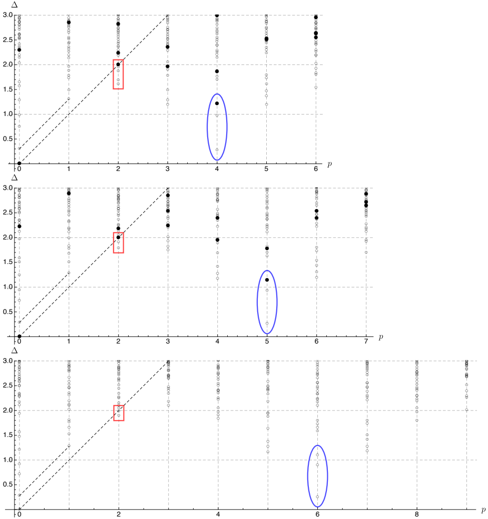


We don't have $\boldsymbol{\rho}$ measurements yet.

$L = 12$

$L = 15$

$L = 18$



What are the **operators in blue**, carrying momentum $p = L/3$?

An operator with momentum p gets the phase

$$e^{2\pi i p/L}$$

under a shift of one site on a finite chain of length L.

An operator with momentum $p = L/3$ gets the phase

$$e^{2\pi i/3}$$

under a shift of one site. We interpret it as follows:

Shifts of sites generate translations in the continuum limit, and

a shift of one site generates a $\mathbb{Z}_3$ symmetry in the continuum limit.

Similar phenomena are common in spin chain realizations of CFTs.

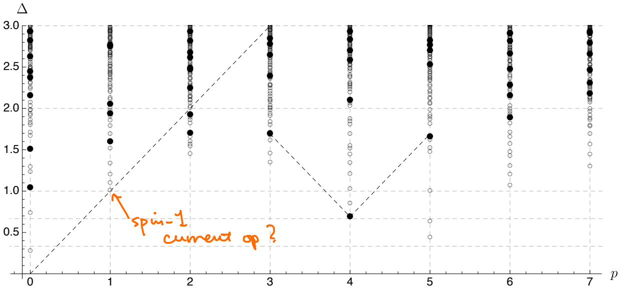
(Note that this shift-$\mathbb{Z}_3$ is in addition to the Haagerup symmetry.)

So, chains of length $\boldsymbol{L = 3N}$ are in the untwisted sector,
but chains of length $\boldsymbol{L = 3N \pm 1}$ are in the $\mathbb{Z}_3$-twisted sector.

This is why I was showing only $\boldsymbol{L = 12}$, $\boldsymbol{15}$ and $\boldsymbol{18}$ so far.

The energy spectra at $\boldsymbol{L = 3N \pm 1}$ look completely different.

$L = 14$



It looks like that there is a spin-1 current of dimension 1...

If this is really the case, the shift-$\mathbb{Z}_3$ orbifold of
our Haagerup ‘anyon chain’ in the continuum limit

- has $c \sim 2$
- has two pairs of left- and right-moving spin-1 currents.

This almost forces it to be a free sigma model on T^2.

Maybe we can construct it analytically?

Summary

Fusion category symmetries generalize finite group symmetries.

Haagerup fusion categories are particularly interesting.

There seems to be (at least) a CFT with this symmetry, around $c \sim 2$.

Personal acknowledgments

I first heard of paragroups and the Haagerup subfactor
in [an article Kawahigashi-sensei wrote in 1996]
in a popular mathematics magazine 数理科学.

I am very happy that I have finally something to say about it.

This is also my first paper with **Masaki Tezuka**,
who I first met I think also in 1996.

(He is a cond-mat theorist, and I asked him to join the collaboration to educate us about DMRG and exact diagonalization, with which the rest of us string theorists are totally unfamiliar.)

I am also quite happy about this.

I first met **Ying-Hsuan Lin** in person on May 2019, when he visited IPMU and gave [this nice talk.]

He was already interested in fusion category symmetries; I convinced him of the importance of Haagerup fusion categories at that time.

Since then he has been working tirelessly with his collaborator **Tzu-Chen Huang** on Haagerup fusion categories, not just on this particular project. I cannot thank them enough.

Kantaro Ohmori is a former student and a long-term collaborator of mine. He gave [an extremely nice review talk, February 2021] on [Aasen-Fendley-Mong 2008.08598] in a workshop also organized by Kawahigashi-sensei.

Listening to his talk, I was convinced that the time was ripe to look for a Haagerup CFT. It took us about half a year since then.