Master's Thesis

Seiberg-Witten Theory and Instanton Counting

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Abstract

We review recent works on the instanton calculation of prepotentials for Yang-Mills theory in four and five dimensions with eight supersymmetries. We firstly review relevant prerequisites, specifically the Atiyah-Drinfeld-Hitchin-Manin construction of instantons and the localization formula of Atiyah-Bott-Lefschetz. We then move on to discuss the celebrated works initiated by Seiberg and Witten, which used strong coupling arguments to determine the low energy prepotential. Finally, combining the knowledge obtained in the two preparatory chapters, we compute the low energy prepotential by direct instanton calculation. An extension of these developments by the author is also briefly reported.

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Chapter 0

Introduction

Dynamical properties of non-ableian gauge field is a notoriously difficult problem. The one-loop beta function is negative. This has virtues in that it can explain the asymptotic freedom of the strong interaction. However, the very same property leads to the conclusion that their low energy spectrum and interactions cannot be studied perturbatively. Experimentally, we know that non-abelian gauge fields confine themselves. This is the inevitable fact if one accepts that quantum chromodynamics(QCD) describes the strong interaction. Hence, we can say that theoretical understanding of the confinement is inarguably one of the most important and most challenging problems for theoretical physicists. Although many theoretical scenarios have emerged over the years (*e.g.* an article [1] by A. M. Polyakov and articles [2, 3, 4] by G. 't Hooft), they did not lead to a satisfactory answer until quite recently. The confinement of pure QCD has rejected all the attempts by theoreticians to date. The reader may know that a one-million-dollar prize has been set up for this problem by Clay Research Institute. No one has yet obtained that prize.

Then, what will be a good way to approach the non-perturbative dynamics of non-abelian gauge fields? Classically, vacua of Yang-Mills equation is labeled by an integer and there are potential barriers of finite height between them. We know that tunnelling paths connecting different vacua are non-perturbative objects. Semi-classical WKB analysis around the action-minimizing configuration would be a natural first try. Configurations which minimize the action satisfies a so-called (anti-)self-dual equation and they are collectively known as *instantons.* For the bosonic case however, the Gaussian correction around the instantons are very complicated and moreover they are divergent in the infrared. They were still too difficult. Here two things come to the rescue.

First is the complete classification and construction of multi-instantons by Atiyah, Drinfeld, Hitchin and Manin[5]. This means that we at least know where to do the integral of the semi-classical approximation. In view of the fact that the (anti-)self-dual equation is still a highly nonlinear differential equation, it is astonishing that they solved the equation at such an early stage of the development of the subject as the whole.

Second is the incorporation of supersymmetry. For models with supersymmetry, bosonic and fermionic contributions to the correction tend to cancel against each other. Hence, we can in principle write down explicitly the integrand to integrate over. This should enable us to calculate the instanton contribution to the dynamics, and indeed in the last decade we saw a steady development along these directions (see for example the great review [6] by M. Shifman and A. Vainshtein.) Moreover, for supersymmetric theories we have another means of investigating the low energy dynamics. That is the use of holomorphy. Namely, since many parts of the Lagrangian of a supersymmetric theory is holomorphic functions of parameters and fields, brief understanding of the neighborhoods of their poles suffices to determine the complete results (one of good reviews is the lecture note [7] by M. Peskin). For example, the low energy prepotential, which governs the lowest derivative part of the dynamics of $\mathcal{N} = 2$ supersymmetric gauge theory, was determined in this way by N. Seiberg and E. Witten[8] in 1994. Hence we have quantities to check the instanton calculation against.

Another big branch of the study of instantons is the application to mathematics. S. K. Donaldson utilized the moduli space of instantons to construct new diffeomorphism invariants for four dimensional manifolds and proved many surprising properties. To state his result in a word, one may say that it is the study of cohomology of the instanton moduli on four-manifolds, while the classical invariants were cohomology of the four manifolds themselves. E. Witten, lead by the suggestions by M. Atiyah, showed in his classic work[9] that Donaldson's new invariants are none other than the correlation functions of certain topologically twisted version of $\mathcal{N} = 2$ supersymmetric gauge theories. If one turns around the argument, one can say that certain quantities in $\mathcal{N} = 2$ supersymmetric gauge theories are determined solely by the topology of the moduli space, eliminating the necessity of complicated integral.

What was gradually realized after the turn of the century is that the prepotential itself is one of such topological quantities[10, 11, 12] and that they can be computed using the method of localization. They were lead to these results by examining carefully the instanton action and by showing that they can be cast into a cohomological framework. Finally N. Nekrasov pinpointed the physical mechanism which makes possible the application of the topological method, and immediately wrote down the all-instanton result of instanton calculation[13]. He and his collaborators showed that their results precisely matched with the prepotential obtained from holomorphy[14]. The aim of this master thesis is to report these unifying developments in the last two years, namely the instanton calculation, utilizing localization, of the low energy properties of $\mathcal{N} = 2$ supersymmetric gauge theories.

Organization of the thesis The master thesis is organized as follows. In chapter 1, we recall several basic facts about the instantons in Yang-Mills theory. We treat the ADHM construction of instantons in great detail. Furthermore, we show how this algebraic description of instantons comes from the dynamics of D-branes. We also discuss the effect of non-commutativity and see the disappearance of the small instanton singularity.

In chapter 2, we review the Atiyah-Singer index theorem, its equivariant versions and other localization techniques. We derive them using supersymmetric quantum mechanics. Some application of the theorems is discussed.

In chapter 3, we summarize the celebrated Seiberg-Witten solution of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. After reviewing the generic properties of $\mathcal{N} = 2$ supersymmetric theories in four dimensions, we give a detailed derivation of the solution for gauge group $SU(2)$. Then we see how this result can be concisely summarized using a family of elliptic curves. Next we extend the analysis to $SU(N)$ gauge groups and hyperelliptic curves. We conclude the chapter by examining the weak coupling expansion of the result obtained in the earlier part of the chapter.

Chapter 4 is the main part of the master thesis. We report the recent development in the instanton calculation of prepotential of $\mathcal{N} = 2$ supersymmetric Yang-Mills. First we shortly review the structure of five dimensional supersymmetry which was crucial in the understanding. Second, we move on to explain why the calculation of the prepotential is reduced to the equivariant index of the instanton moduli. We study in detail the fixed points and their contribution to the index, and will see that they are succinctly summarized using Young tableaux. We compare the result so obtained against the weak coupling expansion obtained in chapter 3 and find satisfactory result. Then we study how to extract the hyperelliptic curves from Young tableaux. Finally, we provide a short exposition of the recent work of the author which studies the effect of five-dimensional Chern-Simons terms to the exact effective prepotential.

In chapter 5 we conclude the thesis by summarizing and assessing the future directions. The organization of the thesis is diagrammatically represented in figure 1.

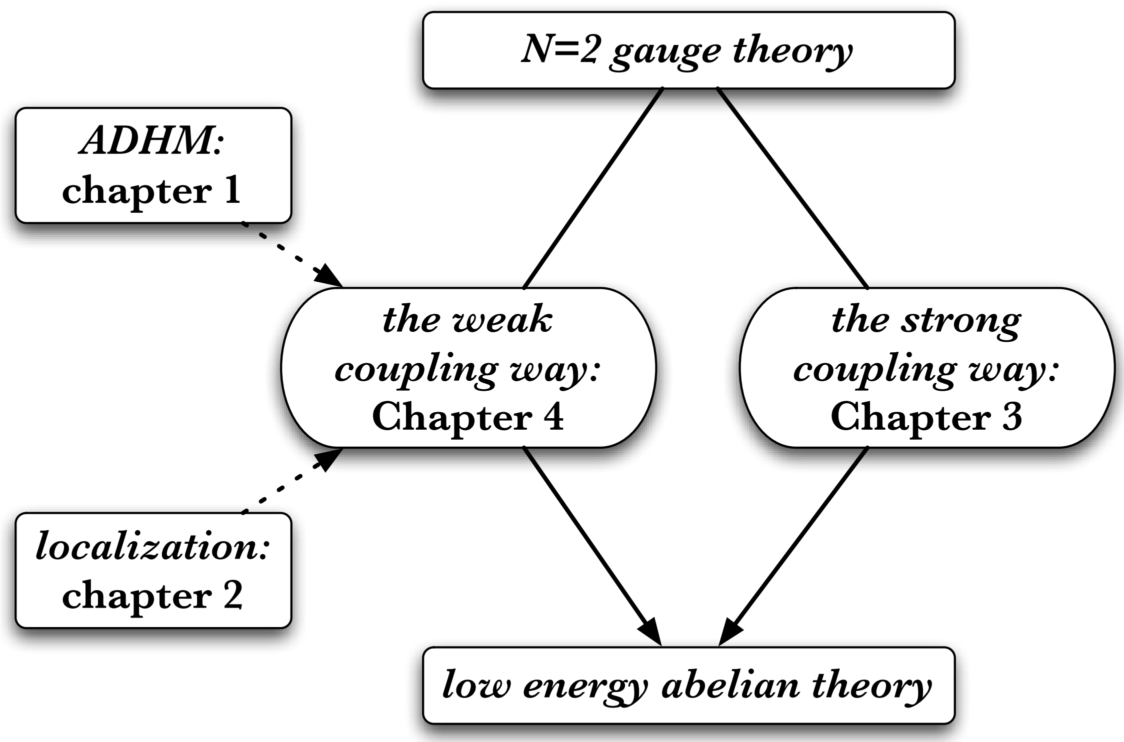


Figure 1: Schematic organization of the thesis

Chapter 1

The instanton moduli

1.1 Geometry of Yang-Mills fields

Firstly, let us recall some basic facts about the geometry of Yang-Mills fields on $\mathbb{R}^4$. The discussions are mainly to show our notations and conventions. We take $\mathbb{R}^4$ to be Euclidean.

For $U(N)$ connections A_μ taking values in anti-hermitean $N \times N$ matrices, the curvature or the field strength $F_{\mu\nu}$ is defined by the commutator of the covariant derivative $D_\mu = \partial_\mu + A_\mu$:

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{1.1}$$

We often denote the curvature using two-form

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{1.2}$$

The Hodge star operation is defined by

$$(*F)_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \tag{1.3}$$

The Yang-Mills action is defined by

$$S = \int d^4x \frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F_{\mu\nu} = \int \frac{1}{g^2} \mathrm{tr} F \wedge *F. \tag{1.4}$$

For the action to be finite, $F_{\mu\nu}$ is to decay sufficiently fast to zero around the spatial infinity, hence the gauge field can be extended to S^4, the one point compactification of $\mathbb{R}^4$. $SU(N)$ bundle on S^4 is obtained by gluing the trivial bundle over each hemisphere along the equator, hence is topologically classified by the homotopy of gluing functions $S^3 \to SU(N)$. As $\pi_3(SU(N)) = \mathbb{Z}$, we see $SU(N)$ gauge bundles are classified by an integer. This is called the instanton number in the physics literature. It is given by the formula

$$\text{instanton number} = \frac{1}{8\pi^2} \int \mathrm{tr} F \wedge F. \tag{1.5}$$

For manifolds other than S^4, this measures the second Chern class of the bundle. We can add to the action

$$+ \frac{i\theta}{8\pi^2} \int \mathrm{tr} F \wedge F. \tag{1.6}$$

The variable θ is normalized so that a k-instanton configuration contributes $k\theta$ to the action. Since k is integer, θ and $\theta + 2\pi$ gives the same contribution under the path integral $\int [dA] e^{-S}$. Hence θ is often called the θ angle.

To carry out semi-classical calculation, it is necessary to find the minimum action configuration in each topological sector. From the equation

$$0 < \int \frac{1}{2g^2} \mathrm{tr}(F - *F) \wedge *(F - *F) \tag{1.7}$$

$$= \frac{1}{g^2} \int \mathrm{tr} F \wedge *F - \frac{1}{g^2} \int \mathrm{tr} F \wedge F \tag{1.8}$$

we find that the Yang-Mills action is bounded by the multiple of instanton number,

$$\frac{1}{g^2} \int F \wedge *F > \pm \frac{8\pi^2}{g^2} \times \text{instanton number} \tag{1.9}$$

and that the bound is attained when the gauge field satisfies

$$F = \pm * F. \tag{1.10}$$

The configuration with + or − sign is called self-dual (SD) or anti-self-dual (ASD), respectively. Hence, the determination of the structure of the solution of this equation is of great physical importance.

There is another physical significance to the anti-self-dual equation. Note that, under the Lorentz group $SO(4) \simeq SU(2) \times SU(2)$, self-dual (anti-self-dual) antisymmetric tensor transform respectively under the representation $(\mathbf{3}, \mathbf{1})$ ($(\mathbf{1}, \mathbf{3})$). In supersymmetric theories, the transformation law of the superpartner λ_α of the gauge field is given by

$$\delta \lambda_\alpha = \epsilon^\beta F_{\alpha\beta} + \cdots \tag{1.11}$$

and

$$\delta \lambda_{\dot\alpha} = \epsilon^{\dot\beta} F_{\dot\alpha\dot\beta} + \cdots . \tag{1.12}$$

This means that when the self-dual part of the curvature $F_{\alpha\beta}$ vanishes, one-half of the original supersymmetry is preserved in the gauge field background. Often, the calculation of various quantity protected by supersymmetry reduces to the analysis of the neighborhood around the configuration with some unbroken supersymmetry. This observation also tells us the importance of the (anti-)self-dual equation.

These developments in physics around nineteen seventies triggered the interest in the mathematical community, and in 1978 Atiyah, Drinfeld, Hitchin and Manin succeeded in the explicit determination of all anti-self-dual connections on S^4. Their method was that of twistor theory and of complex algebraic geometry, and not of the everyday language to many of the physicists. Fortunately, Corrigan and Goddard[16] have found a differential geometric rephrasing of the findings of ADHM, and the results are very understandable. In the following sections we review the construction of all ASD connections on $\mathbb{R}^4$ by their method.

1.2 the ADHM construction

Atiyah, Drinfeld, Hitchin and Manin[5] realized the instanton moduli as a hyperkähler quotient of a vector space. The element of the vector space is often called the ADHM data. We describe in the first subsection the ADHM data and its hyperkähler quotient. Secondly we describe how to construct ASD connection from the ADHM data. Next we discuss briefly the construction of the Dirac zero modes on the ASD connection.

1.2.1 ADHM data and hyperkähler quotient construction

Denote by V and W hermitean complex vector space with dimensions k and N, respectively and denote by

$$\mathbb{X} = (V^* \otimes V) \oplus (V^* \otimes V) \oplus (W^* \otimes V) \oplus (V^* \otimes W). \tag{1.13}$$

Elements of $\mathbb{X}$ are denoted by

$$X = (B_1, B_2, I, J), \tag{1.14}$$

with

$$B_1, B_2 : V \to V, \quad I : W \to V, \quad J : V \to W. \tag{1.15}$$

We can endow the space X by an anti-linear involution

$$J : (B_1, B_2, I, J) \mapsto (B_2^\dagger, -B_1^\dagger, J^\dagger, -I^\dagger) \tag{1.16}$$

Hence, the space $\mathbb{X}$ is naturally a flat, hyperkähler space. $\mathbb{X}$ has a natural action of $U(k)$ and $U(N)$ inherited from the action of these groups to V and W. Moreover, this group action respects the hyperkähler structure. The three moment maps for $U(k)$ action,

$$\mu_i = (\mu_\mathbb{R}, \mathrm{Re}\mu_\mathbb{C}, \mathrm{Im}\mu_\mathbb{C}) : \mathbb{X} \to u(k)^* \otimes \mathbb{R}^3 \tag{1.17}$$

i.e. the Hamiltonian generating the group action on $\mathbb{X}$ with respect to the three hyperkähler forms, are given by

$$\mu_\mathbb{R} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J \tag{1.18}$$
$$\mu_\mathbb{C} = [B_1, B_2] + IJ \tag{1.19}$$

The hyperkähler quotient of $\mathbb{X}$ by $U(k)$ action is defined by

$$\mathbb{X}///U(k) \equiv \mu^{-1}\iota(\zeta^i)/U(k) \tag{1.20}$$

where $\iota : \mathbb{R} \simeq u(1)^* \hookrightarrow u(k)^*$ is the canonical inclusion.

Atiyah, Drinfeld, Hitchin and Manin identified the manifold $M_{n,k} = \mu^{-1}\iota(0)/U(k)$ and the k-instanton moduli of $U(N)$ connection. Let us next see the precise correspondence between the two.

Before going to the next section, we note that the anti-self-dual equation itself can be considered as a hyperkähler reduction in a infinite-dimensional setup. Indeed, consider the space of connections $\mathscr{A}$ on a n-dimensional vector bundle on $\mathbb{R}^4$. Tangent space of $\mathscr{A}$ at a gauge field configuration A_μ is the space of $u(n)$ valued one-forms. One can introduce to this space a metric using

$$(\alpha, \beta) = -\int_M \mathrm{tr}(\alpha \wedge *\beta). \tag{1.21}$$

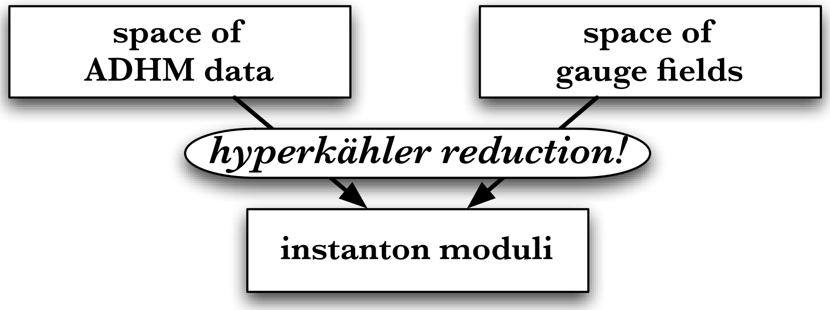


Figure 1.1: Two hyperkähler reductions

Hyperkähler structure on M makes $\mathscr{A}$ a flat, infinite dimensional hyperkähler space. Standard action of gauge group $\mathscr{G}$ = space of maps $\mathbb{R}^4 \to U(N)$ on $\mathscr{A}$ respects the hyperkähler structure, hence one can consider the hyperkähler quotient

$$\mathscr{A}///\mathscr{G}. \tag{1.22}$$

The defining equation can be determined by calculating the moment maps. The moment maps are functions valued in the dual of Lie algebra of $\mathscr{G}$, that is, they determine for each gauge field configuration $u(n)$ adjoint valued function on $\mathbb{R}^4$. The result turns out to be

$$\omega^x \wedge F. \tag{1.23}$$

where ω^x is the three self-dual 2-forms which gives the hyperkähler structure on M. These relations are summarized in figure 1.1.

1.2.2 ADHM data to ASD connection

Construction and anti-self-duality

Given an ADHM data $X = (B_1, B_2, I, J) \in \mathbb{X}$ satisfying $\mu_\mathbb{R} = \mu_\mathbb{C} = 0$, let us construct an ASD connection over $\mathbb{R}^4$. Consider a linear operator depending on $x = (z_1, z_2) \in \mathbb{R}^4 \simeq \mathbb{C}^2$:

$$\nabla^\dagger(x) = \begin{pmatrix} I & -(B_2 - z_2) & B_1 - z_1 \\ J^\dagger & (B_1 - z_1)^\dagger & (B_2 - z_2)^\dagger \end{pmatrix} : \begin{matrix} W \\ \oplus \\ S^- \otimes V \end{matrix} \to S^+ \otimes V, \tag{1.24}$$

where we denoted the two-dimensional representation of positive (negative) chirality spinors respectively by $S^{+(-)}$. A short calculation reveals that $\nabla^\dagger(x)\nabla(x)$ acts on $S^+ \otimes V$ as

$$\nabla^\dagger(x)\nabla(x) = \mathrm{id}_{S^+} \otimes \Box(x) \tag{1.25}$$

where

$$\Box(x) = (B_1 - z_1)(B_1 - z_1)^\dagger + (B_2 - z_2)(B_2 - z_2)^\dagger + II^\dagger \tag{1.26}$$
$$= (B_1 - z_1)^\dagger(B_1 - z_1) + (B_2 - z_2)^\dagger(B_2 - z_2) + J^\dagger J. \tag{1.27}$$

This means that generically the map $\nabla^\dagger(x)$ is surjective, hence $\mathrm{Ker}\nabla^\dagger(x)$ determines a N dimenional vector bundle over $\mathbb{R}^4$. From the hermitian metric of $W \oplus V \oplus V$, a natural $U(N)$ connection is defined on the bundle. Its curvature is anti-self-dual. Although this fact can be derived by direct calculation, we show this by a somewhat longer argument. The method will be necessary later in this master thesis, in chapter 4 .

Firstly, recall basic facts on holomorphic bundles. The aim is to show that the natural unitary connection on a holomorphic bundle has its curvature only in $(1,1)$ component. Let us start by giving some definitions. A holomorphic bundle is a bundle with holomorphic transition functions. A hermitian bundle is a holomorphic bundle with a hermitian metric $(\cdot,\cdot)$. A connection on a holomorphic bundle is holomorphic if represented as a one-form in a patch it is purely of type $(1,0)$. Let us find the condition when a holomorphic connection is simultaneously unitary. For clarifying this, take a holomorphic basis $e_1,\ldots,e_n$ in a patch, and write $h_{i\bar{j}} = (e_i, e_j)$. Then from the Leibnitz rule

$$\partial h + \bar{\partial} h = h_{i\bar{j}} A + A^\dagger h_{i\bar{j}} \tag{1.28}$$

Comparing the type of differential forms, the connection is uniquely determined to be $A = h^{-1}\partial h$. One may say that the connection is 'holomorphically pure gauge'. From this we can easily verify that the curvature has no $(2,0)$ part.

Secondly, let us express $\mathrm{Ker}\nabla^\dagger(x)$ by a cohomology of a complex:

$$V \xrightarrow{\sigma(x)} \begin{matrix} W \\ \oplus \\ S^- \otimes V \end{matrix} \xrightarrow{\tau(x)} V \tag{1.29}$$

where

$$\sigma = \begin{pmatrix} J \\ B_1 - z_1 \\ B_2 - z_2 \end{pmatrix}, \qquad \tau = \begin{pmatrix} I & -(B_2 - z_2) & B_1 - z_1 \end{pmatrix} \tag{1.30}$$

That this forms a complex, that is $\tau(x)\sigma(x) = 0$, follows from a part of ADHM constraints $\mu_{\mathbb{C}} = 0$. Moreover, $\Box(x)$ appeared in (1.27) is

$$\Box(x) = \sigma(x)^\dagger \sigma(x) = \tau(x)\tau(x)^\dagger. \tag{1.31}$$

We will need this relation later.

The cohomology $\mathrm{Ker}\tau/\mathrm{Im}\sigma$ can be identified with $\mathrm{Ker}\nabla^\dagger(x)$ using the metric in $W \oplus V \oplus V$, since

$$\nabla^\dagger(x) = \begin{pmatrix} \tau(x) \\ \sigma^\dagger(x) \end{pmatrix} \tag{1.32}$$

In this representation, it is obvious that the fiber at x, $\mathrm{Ker}\tau/\mathrm{Im}\sigma$ varies holomorphically in $x = (z_1, z_2)$. We saw that a generic consequence of this holomorphy is that the curvature is of type $(1,1)$.

Noticing that $\mathbb{R}^4$ can be given a family of complex structures parametrized by S^2 and the above arguments can be done for every one of them, we have shown the curvature of the bundle $\mathrm{Ker}\nabla^\dagger(x)$ is of type $(1,1)$ in every complex structure. This proves that the curvature is anti-self-dual.

Instanton number

We can also check that the instanton number of the connection is k. This can be verified easily by considering the orthogonal complement of $\mathrm{Ker}\nabla^\dagger(x)$. From the property

$$(a, \nabla^\dagger(x) b) = (\nabla(x) a, b) \tag{1.33}$$

for $a \in V \oplus V$ and $b \in W \oplus V \oplus V$, the orthogonal complement can be identified as $\mathrm{Im}\nabla(x)$. It is essentially k copies of two dimensional subbundle in a four dimensional trivial bundle determined by the map

$$\iota : \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \tag{1.34}$$

since the second Chern class only depends on the asymptotic behavior around infinity of the bundle considered. Hence the bundle $\mathrm{Im}\nabla(x)$ has k times the second Chern class of the bundle $\mathrm{Im}\iota$. The second Chern class of $\mathrm{Im}\iota$ itself can be obtained by studying the transition function at the equator $|x|^2 = 1$ for the bundle. This is the identity map from $|x|^2 = 1 \sim SU(2)$ to $SU(2)$. Hence its instanton number is one, more or less by definition. Combining all this, we see that the bundle $\mathrm{Ker}\nabla^\dagger(x)$ has instanton number $-k$ as desired.

Dirac zero modes on the ASD connection

From the Atiyah-Singer index theorem, we expect the existence of at least k zero modes of fermions coupled to the k-instanton gauge field in the fundamental representation. In fact, there is a vanishing theorem which guarantees the absence of positive chirality zero modes in ASD background. Let us first see how this theorem follows[17].

Let ψ be a solution to the equation

$$\not{D}\psi = 0, \tag{1.35}$$

$$\gamma_5 \psi = \psi. \tag{1.36}$$

We want to show such a ψ is everywhere zero, once square integrability is imposed. First, further applying $\not{D}$ to the left hand side of (1.35), we obtain

$$0 = \not{D}\not{D}\psi = (D_\mu D_\mu + \frac{1}{2} F_{\mu\nu} \gamma^\mu \gamma^\nu)\psi. \tag{1.37}$$

We have used the Lichnerowicz formula. The second term in the above equation vanishes, because the positive chirality of ψ means that it is in the $(\frac{1}{2}, 0)$ representation, which in turn shows that it cannot be multiplied by an ASD curvature on in the $(0, 1)$ representation. Hence we obtained $D_\mu D_\mu \psi = 0$. Thus we see that

$$||D_\mu \phi||^2 = \int d^4 x D_\mu \psi D_\mu \psi = -\int d^4 x \bar{\psi} D_\mu D_\mu \psi = 0. \tag{1.38}$$

This tells us that the ψ is covariantly constant, which is never square integrable unless it is zero everywhere. Q.E.D.

With this vanishing theorem, we know there must be k fundamental zero modes in the ASD background. Indeed, we can construct them from the ADHM data. Zero modes are of negative chirality from the discussion above. The k solutions are given by composition of maps

$$\psi(x) = P(x) \circ \epsilon \circ \square^{-1} : V \to S^- \otimes \mathrm{Ker}\nabla^\dagger(x) \tag{1.39}$$

where

$$\epsilon : V \hookrightarrow S^- W \oplus (\mathbf{1} \oplus \mathbf{3}) \otimes V \sim S^- \otimes (W \oplus S^- \otimes V) \tag{1.40}$$

is the inclusion onto the direct summand and

$$P(x) : W \oplus S^- \otimes V \to \mathrm{Ker}\nabla(x)^\dagger \tag{1.41}$$

is the projection onto the kernel.

Take the basis where the chiral Dirac operator $\not{D} = D_\mu \sigma^\mu$ is given in the complex coordinates by

$$\begin{pmatrix} -D_2 & D_1 \\ \bar{D}_1 & \bar{D}_2 \end{pmatrix}. \tag{1.42}$$

Decomposed into components, $\psi(x)$ becomes

$$\psi_{\dot{1}}(x) = P(x) \begin{pmatrix} 0 \\ \square(x)^{-1} \\ 0 \end{pmatrix}, \qquad \psi_{\dot{2}}(x) = P(x) \begin{pmatrix} 0 \\ 0 \\ \square(x)^{-1} \end{pmatrix}. \tag{1.43}$$

That this gives the solutions to the Dirac equation can be shown with the following elementarycalculation. For example, let us derive $-P(x)\partial_2 P(x)\psi_1(x) + P(x)\partial_1 P(x)\psi_2(x) = 0$. One useful fact is that since the projector is

$$P = 1 - \sigma\square^{-1}\sigma^\dagger - \tau^\dagger\square^{-1}\tau, \tag{1.44}$$

and since $P\sigma = P\tau^\dagger = 0$ from the definition, the derivative of P under the action of P drastically simplifies to

$$P(\partial_i P) = -(\partial_i\sigma)\square^{-1}\sigma^\dagger. \tag{1.45}$$

Secondly, $\partial_i\square^{-1}$ can be similarly simplified to

$$\partial_i\square^{-1} = -\square^{-1}(\partial_i(\sigma^\dagger\sigma))\square^{-1} = -\square^{-1}\sigma^\dagger(\partial_i\sigma)\square^{-1}. \tag{1.46}$$

Combining these, we obtain

$$-P\partial_2(P\psi_1) + P(\partial_1 P\psi_2) = \begin{pmatrix} 0 \\ -(B_2 - z_2)^\dagger\square^{-1} \\ (B_1 - z_1)^\dagger\square^{-1} \end{pmatrix} - \begin{pmatrix} 0 \\ -(B_2 - z_2)^\dagger\square^{-1} \\ (B_1 - z_1)^\dagger\square^{-1} \end{pmatrix} = 0. \tag{1.47}$$

The other equation, $P(x)\bar{\partial}_1 P(x)\psi_1(x) + P(x)\bar{\partial}_2 P(x)\psi_2(x) = 0$, can be proved by the same way.

1.3 Extension to the non-commutative space

We have seen in the previous section that the hyperkähler quotient of a vector space $\mathbb{X}$ at level zero, $\mu^{-1}\iota(0)/U(k)$ coincides to the moduli of ASD instantons. Then it is natural to ask what the quotient at level other than zero describes. An answer has been long known to mathematicians. It describes the moduli of so-called framed torion-free sheaves on $\mathbb{CP}^2$. The extra information arising from the change in the level is carried on the line at infinity. A framed torsion-free sheaf on $\mathbb{C}^2 \sim \mathbb{CP}^2 \setminus l_\infty$ is essentially the same objects as a ASD connection, just phrased in fancy mathematical terms.

A more physical answer is obtained by Nekrasov and Schwarz [18]. They realized that the quotient at non-zero level corresponds to ASD instantons on a non-commutative $\mathbb{R}^4$. We review their construction in this section. For a more detailed review, we refer the reader to [19].

1.3.1 Gauge field on Non-commutative $\mathbb{R}^4$

Let us first reflect a bit on properties we need to define a quantum field theory. We may take the path integral formalism as the basic framework. Then, we need the space of functions to integrate over and the action functional to be integrated. Noticing that the spacetime manifold itself does not appear in this formulation, we may consider replacing the space of functions $\mathscr{A}$ by some other algebra $\mathscr{A}'$ and the action functional by a function on the algebra $\mathscr{A}'$. A most modest modification seems to be deformation of the product structure introduced on the space of functions

$$(f \cdot g)(x) = f(x)g(x) \Longrightarrow (f \star g)(x) = \text{some other operation.} \tag{1.48}$$

We consider the algebra with such a modified product structure as the function space of a 'generalized' version of a manifold. We call the manifold non-commutative when the $\star$ product is non-commutative.

For definiteness, let us take $\mathbb{R}^4$ as the base space and modify the product structure to

$$[x_i, x_j]_\star \equiv x_i \star x_j - x_j \star x_i = i\theta_{ij} \tag{1.49}$$

using an real anti-symmetric matrix θ. The underlying space for this $\star$ product is called the non-commutative $\mathbb{R}^4$. The construction may sound very esoteric, but we will see in the following sections that the non-commutative $\mathbb{R}^4$ naturally arise as a particular kind of closed string background. Furthermore, from a more pragmatic point of view, we will see that this non-commutative deformation can effectively be utilized as an ultraviolet cutoff. Let us pursue the subject for itself for now.

We have seen that the non-commutative space is a 'space' such that its function space is non-commutative. Denote the algebra of functions by $\mathscr{A}$. To consider gauge theory on the non-commutative space, we need to first find how to represent vector bundles on such a space. In order to gain insight, we first rephrase the facts on vector bundles on ordinary commutative space by the language of the algebra $\mathscr{A}$. Then it is possible to generalize the construction to the non-commutative case.

Bundles in the language of algebras Consider a complex n-dimensional trivial bundle $\mathbb{C}^n \times M$ over M. The space of sections is the direct sum of n copies of the space of functions on M, that is

$$\Gamma(\mathbb{C}^n \times M) = \underbrace{\mathscr{A} \oplus \mathscr{A} \cdots \oplus \mathscr{A}}_{n \text{ times}} = \mathbb{C}^N \otimes \mathscr{A} \tag{1.50}$$

We have a natural action of algebra $\mathscr{A}$ on this space by left multiplication. We call a vector space with an action of algebra $\mathscr{A}$ as an $\mathscr{A}$-module. Hence, the space of sections of trivial bundles form an $\mathscr{A}$-module. An $\mathscr{A}$-module with the form $\underbrace{\mathscr{A} \oplus \mathscr{A} \cdots \oplus \mathscr{A}}_{n \text{ times}}$ is called a free module. Similarly for any bundle E over M, one can multiply its section by a function on M, and the result is again a section of E. This means the space of sections of any bundle forms an $\mathscr{A}$-module. We can see the Whitney sum and product operations of bundles correspond to the direct sum and the tensor product over $\mathscr{A}$. The sections of a dual bundle forms the dual of modules, $\Gamma(E^*) = (\Gamma(E))^*$.

Not all of $\mathscr{A}$-modules appear as a space of sections of some bundle, however. It can be shown that for any complex n-dimensional vector bundles E we can find another bundle F such that $E \oplus F = \mathbb{C}^N \times M$ for some N (for a readable proof, see [20]). We give a brief proof here. Consider a good covering U_i ($i = 1, 2, \ldots, r$) of M, and using the partition of unity define functions ψ_i with properties

$$\psi_i \geq 0 \text{ on } M, \qquad \psi_i > 0 \text{ on } U_i, \qquad \sum_i \psi_i(x)^2 = 1 \text{ on } M. \tag{1.51}$$

Denote by g_{ij} the transition functions in $U(n)$ between patches U_i and U_j. Then the original bundle E is isomorphic to the subbundle of $\mathbb{C}^{nr} \times M$ cut out by the projector $p_{ij} : M \to \mathrm{Hom}(\mathbb{C}^{nr}, \mathbb{C}^{nr})$

$$p_{ij} = \begin{cases} \psi_i g_{ij} \psi_j & \text{on} \quad U_i \cap U_j \\ 0 & \text{outside} \quad U_i \cap U_j \end{cases} \tag{1.52}$$

Q.E.D. Translated to a language of algebras, this property can be phrased that for the $\mathscr{A}$-module L of sections on E one can find another module L' such that

$$L \oplus L' = \mathbb{C}^N \otimes \mathscr{A} \tag{1.53}$$

for some N. Such modules are called finite projective. An important theorem of Serre and Swan is that all projective modules of $\mathscr{A}$ arise that way. Indeed, consider a vector space

$$E_p \equiv L/m_p L \qquad \text{where } m_p \equiv \{f \in \mathscr{A} | f(p) = 0\} \tag{1.54}$$

for any such projective module L and a point p in M. This can be shown to be of finite dimension and forms a vector bundle over M.

Let us proceed to rephrase connections in terms of algebras. The covariant derivative D constructed from the connection maps sections of E to sections of $E \otimes TM$. Further it satisfies the Leibnitz rule. These properties can be readily translated in the language of algebras, as a map from a projective module L

$$D : L \to \Gamma(T^*M) \otimes_{\mathscr{A}} L \tag{1.55}$$

which satisfies the condition

$$D(fa) = (df)a + f(Da). \tag{1.56}$$

Bundles on non-commutative spaces After these warm-ups, we can extend these concepts into the non-commutative case with ease. First we need a non-commutative algebra $\mathscr{A}$ of 'functions of the non-commutative space'. Second we need a projective $\mathscr{A}$ module $\mathscr{B}$ to be identified with 'sections of the cotangent bundle', and a map $\partial_\mu : \mathscr{A} \to \mathscr{B}$ satisfying the Leibnitz rule, Then, a bundle on the non-commutative space is a finite projective $\mathscr{A}$-module L, and a connection on it is a map $D : L \to L \otimes \mathscr{B}$ satisfying the Leibnitz rule.

1.3.2 ADHM construction

Using the machinery we developed in the previous section, we can now go on to the construction of ASD instantons on non-commutative $\mathbb{R}^4$. Firstly, we denote the algebra of the functions on non-commutative $\mathbb{R}^4$ as $\mathscr{A}_\theta$ and this is generated by elements $x_1, \dots x_4$ satisfying equation (1.49). We take the underlying set of $\mathscr{A}_\theta$ as the same space of functions on commutative $\mathbb{R}^4$. $\mathscr{A}_\theta$ is endowed with a different product structure $\star$. The sections of the tangent bundle is taken to be just $\mathbb{R}^4 \otimes \mathscr{A}_\theta$, and the derivatives ∂_μ are just the ordinary partial derivatives.

Consider the sequence of maps

$$V \otimes \mathscr{A}_\theta \xrightarrow{\sigma} (W \oplus V \oplus V) \otimes \mathscr{A}_\theta \xrightarrow{\tau} V \otimes \mathscr{A}_\theta \tag{1.57}$$

with

$$\sigma = \begin{pmatrix} J \\ B_1 - z_1 \\ B_2 - z_2 \end{pmatrix}, \qquad \tau = \begin{pmatrix} I & -(B_2 - z_2) & B_1 - z_1 \end{pmatrix} \tag{1.58}$$

imitating the commutative version (1.30). For this to be a chain complex, we need the condition

$$[B_1, B_2] + IJ = [z_1, z_2] \tag{1.59}$$

This is just $\mu_\mathbb{C} = \zeta_\mathbb{C}$, where we denoted $\zeta_\mathbb{C} = [z_1, z_2]$. Here it is important that the commutator of coordinates, z_1 and z_2, is now non-vanishing and gives the magnitude of non-commutativity. Since the module $\mathrm{Ker}\tau/\mathrm{Im}\sigma$ is projective, it determines a bundle over non-commutative $\mathbb{R}^4$. The construction of the induced connection over the quotient bundle, which is reviewed in section 1.2.2, can essentially be carried out, hence we see the curvature is of type $(1, 1)$. This construction goes word-to-word unchanged for any of the three complex structures one can put on $\mathbb{R}^4$ when the ADHM data further satisfy $\mu_\mathbb{R} = \theta_{1\bar{1}} - \theta_{2\bar{2}}$. This shows that the curvature is anti-self-dual.

1.4 Stringy interpretation

We cannot end this chapter without mentioning the beautiful findings of string theorists. They showed that the ADHM construction in its commutative and non-commutative framework can be understood physically using branes in string theory. The appearance of the linear ADHM data was first noted by Witten in [21] where he studied the world sheet theory on the heterotic $SO(32)$ string just before the advent of D-brane revolution. Later, Douglas and Moore[22] found that the ADHM construction can be reproduced in a system with Dp-brane and D$(p+4)$-brane. They further showed that the Kronheimer-Nakajima construction of instantons on the ALE space can be reproduced as well. In view of the S-duality between the type I string theory and the $SO(32)$ heterotic string, Witten's original construction can be thought of as a D1 brane probing the instantons of gauge fields on the ambient 16 D9-branes[23]. Hence we restrict the attention to the D-brane construction.

1.4.1 D-brane construction

Consider a stack of N D7 branes extending in the 0 to 7 direction and another stack of k D3 branes extending in the 0 to 3 direction. The setup is summarized by the table 1.1. Open

	0	1	2	3	4	5	6	7	8	9
D3	–	–	–	–	•	•	•	•	•	•
D7	–	–	–	–	–	–	–	–	•	•

Table 1.1: D3-D7 system. Dot/Dash denotes the object is localized/extended in that particular direction.

strings which have ends on D7-D7, D7-D3, D3-D3 branes give rise to various massless fields. The spectrum can be obtained by quantizing the open strings. The result is summarized in the following data:

Let us view the system from the four dimensional point of view on the D3-branes. When the extra direction $4,5,6,7$ is not compactified, the D7 branes are much heavier or in other words the coupling constant becomes very small. Thus we can neglect their effect and restrict our attention to the fields coming from D3-D3 and D3-D7 strings. The dynamics of their zero-modes is described by the following $\mathcal{N}=2$ supersymmetric Lagrangian

$$\mathscr{L} = \int d^4\theta(\Phi^\dagger\Phi + B_1^\dagger e^V B_1 + B_2^\dagger e^{-V} B_2) + \int d^2\theta \mathrm{tr}(I\Phi J) + \int d^2\theta \mathrm{tr}(B_1[\Phi, B_2]) + \int d^2\theta \mathrm{tr} W_\alpha W^\alpha + c.c. \quad (1.60)$$

From the four dimensional viewpoint, V and Φ forms a $\mathcal{N}=2$ vector multiplet, B_1 and B_2 form an adjoint hypermultiplet, and I and J form N hypermultiplet in the fundamental representation of $U(k)$, with global symmetry $U(N)$. From the string theory point of view, on the other hand, the gauge fields in V are $U(k)$ gauge fields propagating on the stack of k D3 branes, Φ represent the fluctuation of the D3 branes in the 8,9 directions. B_1 and B_2 represent the fluctuation in the 4,5,6,7 directions, i.e. transverse to the D7 branes, and finally I and J comes from open strings stretched between D3 and D7 branes.

The scalar potential can be straightforwardly computable and the result is

$$V = \|[\Phi, B_1]\|^2 + \|[\Phi, B_2]\|^2 + |I\Phi|^2 + |\Phi J|^2 + \left|[\Phi, \Phi^\dagger] + [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J\right|^2 + |[B_1, B_2] + IJ|^2 \quad (1.61)$$

It has, as usual for a supersymmetric theory, manifolds of zero energy configuration, which is called the moduli space of the theory. The moduli can be divided into two. One is the Coulomb branch where the hypermultiplets B_1, B_2, I, J is zero. The other is the Higgs branch where the scalar in the vector multiplet, Φ, is zero and the vacuum expectation value of hypermultiplets are generically nonzero. Let us concentrate on the study of the Higgs branch. The defining equation is

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \quad (1.62)$$

$$[B_1, B_2] + IJ = 0. \quad (1.63)$$

The $U(k)$ local gauge symmetry in the system means that the true moduli is obtained by dividing the space $= 0$ by $U(k)$ action. Surprisingly, this is exactly the same procedure to obtain the instanton moduli $M_{N,k}$ from the linear ADHM data. The natural question is where we should find the anti-self-dual instanton. One thing to note is that $\phi = 0$ in the Higgs branch. This corresponds to the situation where the D3 branes are stuck just on the D7 branes. Let us see in the next subsection how this observation links to the identification of a D3 brane and an instanton inside the D7 branes.

1.4.2 Small instanton singularity

One annoying but important feature of the instantons is the presence of the small instanton singularity. The occurrence of the singularity is most easily seen in the explicit one-instanton solution of $SU(2)$ gauge theory, which is

$$F_{\mu\nu} = \frac{2\eta^{-}_{\mu\nu}\rho^2}{|x_\mu|^2 + \rho^2} \tag{1.64}$$

The parameter ρ is undetermined and in the $\rho \to 0$ limit, the gauge field configuration is almost everywhere pure gauge and the only structure resides very near the center of the instanton x_0. Hence the moduli space has a component which look like $\mathbb{R}^+$ and this becomes non-compact. This reflects the fact that the very theory we are considering, the Yang-Mills theory or the ASD equation has conformal invariance, hence the solution can be made arbitrarily small by an application of scale transformation. We can no longer appeal to conformal symmetry for multi-instantons or instantons on generic four-manifolds but it is known that one out of k in a k-instanton configuration can be made shrunken to zero size. The limit configuration is a point-like instanton superimposed on a $(k-1)$-instanton configuration[24].

Now that we have a D-brane interpretation of the ADHM construction, we have a physical interpretation of these small instanton singularity. Indeed, the D3-D7 system considered above has Coulomb branch in addition to $M_{N,k}$. Moreover, the small instanton singularity is exactly the point where the Coulomb branch touches the Higgs branch. Thus, we can pass continuously from the Higgs branch to the Coulomb branch. As the Coulomb branch with $\phi_i \neq 0$ corresponds to D3 branes not exactly on the D7 branes, this transition is a transition from the ASD instanton of the gauge field on D7 branes to a D3 brane outside the D7 branes. This chain of argument shows that a D3 brane and an instanton in a stack of D7 branes is one and the same thing. This can be further checked against the coupling to the Ramond-Ramond fields.

One of the fundamental properties of D-branes is that they are the sources for the anti-symmetric tensor fields of the type II supergravities. A Dp-brane is the source for $(p+1)$-form field, i.e. it couples through the term

$$\int C^{(p+1)}. \tag{1.65}$$

Consideration of the consistency under the action of T-duality on D-branes and RR fields, the coupling above must be extended to include

$$\int e^F \sum C = \int C^{(p+1)} + \int C^{(p-1)} \wedge F + \int C^{(p-3)} \wedge F \wedge F + \cdots \tag{1.66}$$

From these couplings to the RR fields, we can observe that a D7 brane couples to the four-form field $C^{(4)}$ when $F \wedge F$ is non-zero. As the $F \wedge F$ measures the instanton charge and

	0	1	2	3	4	5	6	7	8	9
D3	–	–	–	–	•	•	•	•	•	•
D7	–	–	–	–	–	–	–	–	•	•
D(−1)	•	•	•	•	•	•	•	•	•	•

Table 1.2: D(−1)-D3-D7 system. Dot/Dash denotes the object is localized/extended in that particular direction.

$C^{(4)}$ is a gauge potential primarily couples to D3-branes, we can say that an instanton in D7 brane 'behaves like' a D3-brane. When the instanton shrinks to zero size and $F \wedge F$ has a delta-function form, the locus M of the instanton core in the D7 brane couples to $C^{(4)}$ through

$$\int_M C^{(4)}, \tag{1.67}$$

just as a D3-brane would couple. All this is consistent and rich.

1.4.3 Seeing the gauge field by another brane

We saw in the last two sections that the ADHM data naturally arise from the D3-D7 system and further saw that this leads to the identification of a pointlike instanton in the D7 worldvolume and a D3 brane. In fact, we can reproduce in a string theory language the construction of the gauge bundle in section 1.2.2. Let us very briefly see how this is done[23].

To measure the gauge field on the D7 branes, we need to insert a probe. We need a very heavy probe. Fundamental strings are not suitable, since the very gauge field we want to measure is one of the oscillation modes of the fundamental strings. Another D-brane suits very well to this end.

Consider a D(−1)-brane in the D3-D7 system, with k D3-branes and N D7-branes. The quantization of open strings result in the following set of states

from D(−1)-D(−1): two complex bosons z_1, z_2
from D3-D(−1): $2k$ complex fermions χ_1, χ_2
from D7-D(−1): N complex fermions λ.

in addition to the states already appeared in (1.60). Calculation of disk three point functions tells us that there are couplings of the form

$$(\bar{\chi}_1, \bar{\chi}_2)\begin{pmatrix} B_1 - z_1 & B_2 - z_2 & I \\ B_2 - z_2 & B_1 - z_1 & J \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \\ \lambda \end{pmatrix}. \tag{1.68}$$

For example, the term $\bar{\chi} I \lambda$ comes from a disk with insertions of (−13), (37), (7−1) vertices. Important point to notice is that the mass term for the fermions is precisely the matrix $\nabla^\dagger(x)$. Hence we see the N massless fermions are the sections of the bundle $\mathrm{Ker}\nabla^\dagger(x)$, that is, they couple to the ASD gauge field.

There is another way to see the ASD gauge field in the D-brane setup[25]. The authors showed that, just as boundary states in closed string Hilbert space describe the back reaction of D-branes to the closed string background, a D-brane inside another brane can be represented by a 'boundary state' $|B\rangle$ in open string Hilbert space, and that the gauge field can be reconstructed from one-point functions

$$\langle V|B\rangle + \langle V|(|B\rangle * |B\rangle) + \langle V|(|B\rangle * |B\rangle * |B\rangle) + \cdots \tag{1.69}$$

This approach is interesting in the light of recent works on closed string boundary state and the $*$ product structure between them[26, 27]. It is the open string version of the higher order backreaction of a D-brane. For more details, we refer the reader to the original article.

1.4.4 B-field, non-commutativity and modified ADHM constraint

We analyzed anti-self-dual connections and the ADHM construction on non-commutative $\mathbb{R}^4$. Although non-commutative spacetimes may seem somewhat mysterious, we now know that they arise naturally within string theory. Originally the appearance of the non-commutative spacetime is noted in [28] in a BFSS/IKKT matrix model setup. Later the appearance was derived using T-duality[29] or using directly the open string calculation[30]. Let us first briefly review the argument in [30].

Consider a string moving in a B-field, with action

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma \left(g_{ij}\partial_a X^i \partial^a X^j - 2\pi i\alpha' B_{ij}\epsilon^{ab}\partial_a X^i \partial_b X^j\right). \tag{1.70}$$

One of the consistent conditions one can impose on the world sheet boundary is

$$g_{ij}\partial_n X^i + 2\pi i\alpha' B_{ij}\partial_t X^j = 0 \quad \text{at the boundary.} \tag{1.71}$$

Let us take the world sheet as the upper half plane $\mathbb{H}$. The propagator can be calculated with the method of images and the result is

$$\langle X^i(\tau)X^j(\tau')\rangle = -\alpha' G^{ij}\log(\tau-\tau')^2 + \frac{i}{2}\theta^{ij}\mathrm{sign}(\tau-\tau') \tag{1.72}$$

where

$$G^{ij} = (g+2\pi\alpha' B)^{-1}|_{\text{sym. part}}, \qquad \theta^{ij} = 2\pi\alpha'(g+2\pi\alpha' B)^{-1}|_{\text{anti-sym. part}}. \tag{1.73}$$

Hence, scattering amplitudes of open strings in the presence of B-field becomes

$$\left\langle e^{ip_i^{(1)}X^i(\tau_1)}\cdots e^{ip_i^{(k)}X^i(\tau_k)}\right\rangle_{G,\theta} = e^{-\frac{i}{4}p_i^{(n)}p_j^m\theta^{ij}\mathrm{sign}(\tau_n-\tau_m)}\left\langle e^{ip_i^{(1)}X^i(\tau_1)}\cdots e^{ip_i^{(k)}X^i(\tau_k)}\right\rangle_{G,\theta=0}. \tag{1.74}$$

A Fourier expansion tells us that the factor

$$e^{-\frac{i}{4}p_i^{(n)}p_j^m\theta^{ij}\mathrm{sign}(\tau_n-\tau_m)} \tag{1.75}$$

is exactly what appears in the vertex of the Feynman diagram when one changes the action from

$$\int d^dx\phi_1(x)\phi_2(x)\cdots\phi_k(x) \qquad \text{to} \qquad \int d^dx\phi_1(x)\star\phi_2(x)\star\cdots\star\phi_k(x). \tag{1.76}$$

	0	1	2	3	4	5	6	7	8	9
D3	–	–	–	–	•	•	•	•	•	•
D7	–	–	–	–	–	–	–	–	•	•
B	•	•	•	•	–	–	–	–	•	•

Table 1.3: D3-D7 system with B-field. Dot/Dash denotes the object is localized/extended in that particular direction. For B-field, Dash denotes that the component along that direction is non-zero.

These calculation reveals that the theory on the D-brane can be thought of as living in a non-commutative plane with $[x^i, x^j]_\star = i\theta^{ij}$.

We saw that there is non-commutativity in the presence of B-field. Hence, the ASD equation in the presence of B-field should be governed by the modified ADHM equation (1.59). Another stringy miracle is that we can see directly in a D-brane setup that B-field modifies the ADHM equation just as expected[22]. In order to see this, recall the Lagrangian (1.60) of the D3-D7 system. Introduce B-field in the 4,5,6,7 directions, that is transverse to the D3-branes and along the D7-branes. We can check by a disk calculation that there is a coupling in addition to the equation (1.60) which gives

$$\mathscr{L}_B = \int d^4\theta (b_{1\bar{1}} - b_{2\bar{2}}) V + \int d^2\theta b_{12}\Phi + c.c. \tag{1.77}$$

where we denoted the vacuum expectation value of the B-field by b This is the Fayet–Iliopoulos term in $N = 2$ supersymmetry, hence it modifies the scalar potential to

$$V = \cdots + \left|[\Phi, \Phi^\dagger] + [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J - (b_{1\bar{1}} - b_{2\bar{2}})\right|^2 + |[B_1, B_2] + IJ - b_{12}|^2 \tag{1.78}$$

Therefore, the equations describing the Higgs branch change into $\mu_\mathbb{C} = B_{12}$ and $\mu_\mathbb{R} = b_{1\bar{1}} - b_{2\bar{2}}$. This is exactly the equation defining the moduli space of non-commutative instantons.

Chapter 2

The method of localization

2.1 Supersymmetric quantum mechanics and the Witten index

Let us consider a quantum mechanical system with discrete spectrum, with the following properties:

1. there is a conserved quantity $(-)^F$ whose eigenvalues are ± 1,
2. there are conserved operators Q_i anticommuting with each other and anticommuting with $(-)^F$, and
3. the Hamiltonian is expressible as $H = \sum_i Q_i Q_i^\dagger$.

We call such a system a supersymmetric quantum mechanics. We sometimes abbreviate it as a SUSY QM. Consider the following quantity

$$\mathrm{Tr}(-)^F e^{-\beta H}. \tag{2.1}$$

Each eigenspace of H forms a representation space for Q_i. Even dimensional representations are non-trivial. $\mathrm{Tr}(-)^F$ is always zero when traced over such an representation, because a state $|\psi\rangle$with $(-)^F = +1$ is always paired with another state $Q_i|\psi\rangle$ of $(-)^F = -1$ and thus they cancel out. An unpaired state should necessarily form a trivial one-dimensional representation. Such state contributes to the $\mathrm{Tr}(-)^F e^{-\beta H}$. Since the eigenvalue of H is zero on a unpaired state because of the condition (2), we see that the quantity (2.1) is independent of β. This is called the Witten index of the system, which was first introduced in [31]. The relation with mathematics of index theorems is further presented in [32]. It is often denoted by Ind. From the independence on β, the index can be calculated both in the $\beta \to 0$ limit and the $\beta \to \infty$ limit. The equality of two limits often leads to non-trivial mathematical result.

The above argument also shows that the Witten index has a certain kind of stability, that is, it does not change under small perturbations preserving supersymmetry. It is because, since the states should always be paired when the energy of them changes from zero to non-zero or non-zero to zero, they do not contribute to the Witten index.

These indices are sometimes useful also for Hamiltonians with continuum spectrum. For such cases, although the range of the fermionic and the bosonic spectra agree, the density of states are not necessarily equal to each other. Hence the index is no longer independent of β. It is known that much can be learned in that case in spite of the difficulty. Moreover, although the index is stable against small perturbation, it changes its value when a zero energy state comes in or goes out of the Hilbert space when the Hamiltonian is perturbed drastically.

We can also consider the quantity

$$\mathrm{Tr}(-)^F e^{-\beta H} g. \tag{2.2}$$

for a bosonic conserved operator g. This is no longer β-independent for general g and thus it is difficult to study. However, if g commutes with all of the supercharges Q_i, each of the irreducible component of the representation of Q_i becomes the eigenspace of g. Hence, non-zero contribution for (2.2) comes only from trivial, one-dimensional representation of Q_i, that is, zero energy states. This means that the quantity (2.2) is independent of β. The number is called the equivariant index of the system and is denoted by Ind_g. This is generally a complex number, instead of an integer as the ordinary index is. The equivariant index also shares the stability inherent in the ordinary Witten index.

Such a conserved quantity g will form a group G. For a conjugate element in $g, h^{-1}gh \in G$ it is easy to see the equivariant index $\mathrm{Ind}_g = \mathrm{Ind}_{h^{-1}gh}$. Hence the equivariant index gives a character for the group G. It is the character of the representation of G formed by the zero energy states. Thus, the equivariant index is sometimes denoted by $\mathrm{Ind}_G \in R(G)$, where $R(G)$ is the space of representations of G. For Abelian G, Ind_G is considered to take the value in the dual group G^*.

One of the good properties of equivariant indices is that, they sometimes make possible the information of the system even when the naïve index gives infinity or zero. For a symmetry group G and its subgroup H, the relation

$$\mathrm{Ind}_H = \mathrm{Res}^G_H \mathrm{Ind}_G \tag{2.3}$$

holds. The calculation of equivariant indices using localization often simplifies for larger symmetry group G. Hence, it is technically worth while to consider the equivariant index for some larger group even when one wants to know the index for a smaller group.

Before moving to the next section and seeing various examples, we want to make a comment on the case when g do not commute with all of the supercharges, but commutes with some of them, say Q. In that case, the quantity $\mathrm{Tr}(-)^F e^{-\beta H}$ becomes β-dependent. The argument above shows, however, that the trace only receives contributions from states annihilated by Q. This makes the calculation tremendously easier and endows the quantity some stability.

2.2 SUSY QM and the Atiyah-Singer index theorem

Let us move on to the examples. Consider a quantum mechanical particle moving in a spin manifold M. When only a bosonic degree of freedom is present, the Hilbert space is that of square integrable functions on M, $C^2(M)$. It can be made supersymmetric. We take the Hilbert space to be the space of square integrable sections $\Gamma(S)$ of the spin bundle S. The fermion number is defined by the action of $\gamma^5 = \gamma^1\gamma^2\cdots\gamma^d$, the Hamiltonian is the square of the Dirac operator, and the supercharge is the Dirac operator itself. This system satisfies the criteria listed in section 2.1, hence we can consider the index of the system

$$\mathrm{Tr}(-)^F e^{-\beta H} \tag{2.4}$$

and we know this is β independent. In the $\beta \to \infty$ limit, this calculates

$$\mathrm{Tr}_{\text{Dirac zero modes}}\gamma^5, \tag{2.5}$$

i.e. the celebrated Dirac index of the manifold. In physical terms, this number determines the gravitational chiral anomaly of the free fermion system on M.

Let us now compute the index in the opposite limit, $\beta \to 0$[33]. It is useful to introduce a Lagrangian description of the supersymmetric quantum mechanics and to do a path-integral. The Lagrangian is

$$\mathscr{L} = \frac{1}{2}g_{ij}\partial_t x^i \partial_t x^j + \frac{1}{2}g_{ij}\psi^i(\delta^i_j\partial_t + \Gamma^i_{jk}\partial_t x^j)\psi^k \tag{2.6}$$

where ψ^i are real Grassmann variables. We can check this Lagrangian is invariant under the supertransformation

$$\delta x^i = \epsilon\phi^i, \qquad \delta\psi^i = \epsilon\partial_t x^i. \tag{2.7}$$

Canonical quantization of the system reveals that the Hilbert space is the space of sections of the spin bundle. This shows that the index we want to calculate is

$$\mathrm{Tr}(-)^F e^{-\beta H} = \int_{\text{periodic}} [dx][d\psi]\exp(-\int_0^\beta dt\mathscr{L}). \tag{2.8}$$

The periodic boundary condition for the fermion corresponds to the insertion of $(-)^F$ under the trace. In the small β limit, x and ψ can not move very far, hence the configurations with $x = x_0$ and $\psi = \psi_0$ constant dominate the integral. We can approximate the path integral by the Gaussian integration of fluctuations followed by the integration of zero modes. Corrections from higher interaction terms drop out in the limit $\beta \to 0$. The Lagrangian for quadratic fluctuation around x_0, ψ_0 is, after taking Riemann normal coordinate system around x_0,

$$\mathscr{L}^{(2)} = \frac{1}{2}g_{ij}(x_0)\partial_t\xi^i\partial_t\xi^j - \frac{1}{4}R_{ijkl}\xi^i\partial_t\xi^j\psi_0^k\psi_0^l + \frac{i}{2}\eta^a\partial_t\eta^a \tag{2.9}$$

where we denoted the fluctuations by $\xi^i = x^i - x_0^i$ and $\eta^i = \psi^i - \psi_0^i$. The Lagrangian is that of a particle moving inside the magnetic field $R_{ijkl}\psi^k\psi^l/2$. This can be diagonalized with eigenvalues $\pm\theta_1, \ldots, \pm\theta_{d/2}$. Thus, the partition function is

$$\mathrm{Tr}(-)^F e^{-\beta H} = \mathscr{N}\int d^d x_0 d^d\psi_0 \prod_{\alpha=1}^{d/2}\det\begin{pmatrix}\partial_t & \theta_\alpha \\ -\theta_\alpha & \partial_t,\end{pmatrix}^{-1/2} \tag{2.10}$$

where $\mathscr{N}$ is an unknown normalization. The determinant can be calculated either by canonical quantization of a particle under the influence of constant magnetic field, or just using the infinite product representation of the sin function:

$$\begin{pmatrix}\partial_t & \theta \\ -\theta & \partial_t,\end{pmatrix} = \prod_{n\neq 0}(\theta^2 - (2\pi n)^2) = \left(\frac{\sin\theta/2}{\theta/2}\right)^2. \tag{2.11}$$

Here we excluded the constant modes $n = 0$ because they are accounted by the zero mode integral $\int dx_0 d\psi_0$.

Considering the Grassmann integration as the integration of differential forms over the manifold M, we can present the result in a conventional way. The result is,

$$\text{the index} = \int_M \prod_\alpha \frac{x_\alpha/2}{\sinh(x_\alpha/2)}. \tag{2.12}$$

where x_α is the 'eigenvalues' of the curvature two-form $\Omega = R_{ijkl}dx^k dx^l/(4\pi)$. The normalization constant which we will determine in the next section is incorporated in advance. We need to make some comments on the quotation marks around 'eigenvalue'. Although an antisymmetric tensor can be transformed into canonical form, Ω, being a two-form valued in anti-symmetric tensor, cannot be in general made to the canonical form. However, by first expanding the integrand into the polynomials in x, we see that the outcome is a symmetric polynomial, thus they can be expressed as the traces of powers of Ω. The integrand should be taken as such. The differential form in the integrand is often denoted by $\hat{A}(TM)$, and called the A-roof genus of the tangent bundle.

Things become more interesting when we couple the point particle system to external gauge field. Mathematically this means the tensoring of the spin bundle S by the vector bundle E determined by the gauge field. The calculation of the index in the $\beta \to 0$ limit is modified to include the effects of the gauge field strength. Consider the following addition to the Lagrangian (2.6)

$$\mathscr{L}_c = ic_A^\dagger(\delta_A^B\partial_t + iA_{Ai}^B\partial_t x^i)c_B + \frac{1}{2}\psi^i\psi^j F_{ij}{}_A^B c_A^\dagger c_B, \tag{2.13}$$

where $A, B = 1, \ldots, \dim E$ and A_i is the connection on E. Total Lagrangian $\mathscr{L} + \mathscr{L}_c$ is still supersymmetric under the transformation

$$\delta x^i = \epsilon\phi^i, \quad \delta\psi^i = \epsilon\partial_t x^i, \quad \delta c_A = 0. \tag{2.14}$$

The Hilbert space of the system is obtained by the quantization of the fermions c_A, and it leads to $\bigwedge^* E$. Hence to calculate the number of zero-modes of $S \otimes E$, we need to extract the contribution of E to the partition function. The quadratic part of the c fermion is

$$c_A^\dagger\partial_t c_A + \frac{1}{2}\psi_0^i\psi_0^j F_{ij}{}_A^B c_A^\dagger c_B. \tag{2.15}$$

Hence its Hamiltonian is $\frac{1}{2}\psi_0^i\psi_0^j F_{ij} c_A^\dagger c_B$, and can be exponentiated easily. Its trace on E, rather than on all of $\bigwedge^* E$ is

$$ch(E) \equiv \mathrm{Tr} e^{F/2\pi} \tag{2.16}$$

where $F = F_{ij}dx^i dx^j/2$. $ch(E)$ is called the Chern character of the bundle. Incorporating this we obtain the final result

$$\mathrm{Tr}_{\text{zero modes}}\gamma^5 = \int_M \hat{A}(TM) \wedge ch(E). \tag{2.17}$$

2.3 SUSY QM and the Euler number

As a next example, consider a quantum mechanical particle, moving in a manifold M. Here we consider a supersymmetrized system with twice as many supercharges as that treated in the last section. Its Hilbert space is the space $\Omega(M)$ of all square integrable sections of differential forms on M. For spin manifold M, $\Omega(M)$ is the square of the spin bundle $\Omega(M) \sim \Gamma(S \otimes S)$. The Hamiltonian of the system is the Laplace-Beltrami operator $(d + *d*)^2$, and the fermion number is given by $(-)^F = (-)^{\text{degree of differential forms}}$. The operator $d + *d*$ sends boson to fermions and commutes with the Hamiltonian, hence can be regarded as the supercharge. The index of the system,

$$\mathrm{Tr}(-)^F e^{-\beta H}, \tag{2.18}$$

is equal to

$$\mathrm{Tr}_{\text{zero energy states}}(-)^F. \tag{2.19}$$

This is none other than the Euler number $\chi(M)$ of the manifold M, because

$$H^p(M) \simeq \mathrm{Ker}(d + *d*)|_{\Omega^p(M)} \tag{2.20}$$

from the Hodge theorem.

The calculation in the $\beta \to 0$ limit can be done using the result obtained in the last section. Indeed, viewing $\bigwedge^* TM$ as $S \otimes S$ and using the fact that the 'eigenvalues' of the curvature of the spin bundle is given by

$$\pm \frac{x_1}{2} \pm \frac{x_2}{2} \cdots \pm \frac{x_{d/2}}{2}, \tag{2.21}$$

the index is

$$\int_M \hat{A}(TM) \wedge ch(S) = \int_M \prod_\alpha \frac{x_\alpha/2}{\sinh x_\alpha/2} \prod_\alpha \sinh \frac{x_\alpha}{2} \tag{2.22}$$

from the Atiyah-Singer index theorem. This can be further simplified to

$$= \int_M \prod_\alpha x_\alpha = \int_M \mathrm{Pfaff}\frac{R_{ij}}{2\pi} = \int_M e(TM) \tag{2.23}$$

Moreover, this can be utilized in determining the normalization appearing in equation (2.12) for example by taking $M = S^{2n}$, because we know the index should equal to its Euler number, 2.

2.4 SUSY QM and the Lefschetz fixed point theorem

Let us next consider the case where a supersymmetric particle is moving on a manifold M with a vector bundle L which admits a $U(1)$ action $g = e^{iJ}$. The Hilbert space of the supersymmetric quantum mechanical system is the same as in the previous section, and we consider the equivariant index

$$\mathrm{Tr}(-)^F e^{-\beta H} g. \tag{2.24}$$

By assumption the Dirac operator $S \otimes L \to S \otimes L$ commutes with the action of g, hence the quantity (2.24) is independent of β. Taking the limit $\beta \to \infty$, we see that they are equal to

$$\mathrm{Tr}_{\text{zero modes}}(-)^F g, \tag{2.25}$$

i.e. it is the character of the $U(1)$ action of the represention which the harmonic spinors form.

Calculating the equivariant index in the limit $\beta \to 0$ reveals that it can be given as a sum of the contributions from each of the fixed points of g action. Indeed, when $\beta \to 0$ and passing to the path-integral representation, it can be represented by

$$\int [dl] \exp(-\int_0^\beta dt \mathscr{L}) \tag{2.26}$$

where $[dl]$ is functional measure on the space of paths

$$dl : [0, \beta] \to M \text{ with } l(0) = gl(\beta). \tag{2.27}$$

It is easy to see that paths connecting $l(0) \neq l(\beta)$ cost too much action and thus do not contribute to the index.

Let us compute the contribution from each of the fixed point. Decompose the action of g to the tangent space and the fiber of E at the fixed point as

$$iJ\Big|_{TM_p} = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & 0 & \theta_2 & \\ & & -\theta_2 & 0 & \\ & & & & \ddots \end{pmatrix} \quad \text{and} \quad iJ\Big|_{E_p} = \begin{pmatrix} iw_1 & & & \\ & iw_2 & & \\ & & iw_3 & \\ & & & \ddots \end{pmatrix} \tag{2.28}$$

The quantum mechanical system around the fixed point is that of free supersymmetric point particle. First consider the case where E is a trivial one dimensional bundle. In this case, the system breaks up into $d/2$ quantum mechanical systems each coming from the eigenspace of $J\big|_{TM_p}$. The contribution from one plane is

$$\mathrm{Tr}(-)^F e^{-\beta H} e^{iJ}, \tag{2.29}$$

where $H = p_x^2 + p_y^2$ with one massless complex fermion $\{\psi, \psi^\dagger\} = 1$. To tame the continuous spectrum, we artificially give the system mass m. Then the trace is calculable and gives

$$e^{\beta m} \frac{e^{\beta m/2 - i\theta/2} - e^{-\beta m/2 + i\theta/2}}{(1 - e^{-\beta m - i\theta})(1 - e^{-\beta m + i\theta})} \xrightarrow{m \to 0} \frac{1}{e^{i\theta/2} - e^{-i\theta/2}}. \tag{2.30}$$

Collecting all factors, we finally obtain

$$\mathrm{Ind}_g = \sum_{\text{f.p.}} \sum e^{iw_i} \prod_{\alpha=1}^{d/2} \frac{1}{e^{i\theta_\alpha/2} - e^{-i\theta_\alpha/2}}. \tag{2.31}$$

This supersymmetric derivation of the fixed point theorem was long known to experts. Some of the earliest references are [34, 35] in which the authors discuss the fermion quantum number after the dimensional reduction on a manifold which admits a symmetry action.

In order to illustrate the power of the fixed point formula, let us show the Weyl character formula by constructing a representation as a cohomology. Consider a compact semisimple group G and its maximal torus T. The complexified tangent space of a point on G/T is spanned by the roots of G. We can make G/T into a Kähler manifold by choosing the positive roots to span the holomorphic tangent space and the negative roots to span its conjugate. Take a highest weight λ and consider the bundle $E_\lambda = G/T \otimes_T V_\lambda$, where V_λ is one dimensional representation of T with character λ. There is an isomorphism $H^{(0,0)}(E) = R$ between the lowest Dolbeault cohomology and R itself. To see this, first view sections of E as T-equivariant maps from G to V_λ. Using the Peter-Weyl decomposition of functions on G

$$\mathscr{L}^2(G) \sim \sum_{\text{irrep } R} R \otimes R^* \tag{2.32}$$

under left and right $G \times G$ action, holomorphic sections of E are seen to comprise the subspace of $\mathscr{L}^2(G)$

$$\psi_\lambda \otimes R^* \subset \sum R \otimes R^* \sim \mathscr{L}^2(G) \tag{2.33}$$

where ψ_λ is highest weight vector of weight λ. Furthermore, as can be checked by direct calculation, the bundle E is positive. Thus, from the Kodaira vanishing theorem, higher Dolbeault cohomologies of E all vanish. Hence the character can be identified with the equivariant index

$$\chi_R(g) = \sum_i (-)^i \mathrm{Tr}_{H^{(0,i)}(E)} g = \mathrm{Tr}(-)^F e^{-\beta H} g. \tag{2.34}$$

Here the supersymmetric quantum mechanics is the one which is realized on a Kähler manifold and has $\Omega^{(*,*)}(TM)$ as its Hilbert space. The supercharges are ∂ and $\bar\partial$. The fermion number we are considering is the one for $\bar\partial$. Nevertheless, the fixed point theorem can be similarly proved for this case and results in

$$\mathrm{Ind}_g = \sum_{\text{f.p.}} \sum e^{iw_i} \prod_{\alpha=1}^{d/2} \frac{1}{1 - e^{-i\theta_\alpha}}, \tag{2.35}$$

where $d/2$ counts the complex dimension of the Kähler manifold.

Let us apply the formula to the present case. The fact that a point $p \in G/T$ is fixed by the left T action translates into $p \in W$ where W is the Weyl group of G. The action of g on $TM|_p$ and $E|_p$ is immediately calculable, as this is the definition of weights of the representation! The result is

$$\chi_R(e^{iJ}) = \sum_{p \in W} \frac{\mathrm{sign}(p) e^{i\langle \lambda, pJ\rangle}}{\prod_{r \in R^+} (1 - e^{-i\langle r, J\rangle})} = \sum_{p \in W} \frac{\mathrm{sign}(p) e^{i\langle \lambda + \rho, pJ\rangle}}{\prod_{r \in R^+} (e^{i\langle r, J\rangle/2} - e^{-i\langle r, J\rangle/2})} \tag{2.36}$$

where R^+ is the set of positive roots and $\rho = \sum_{r \in R^+} r/2$. This is the character formula of Weyl.

Let us see explicitly the case $G = SU(2)$. Here $G/T = \mathbb{CP}^1$ and the bundle E_n is what is usually denoted as $\mathscr{O}(n)$. We know well that $\mathscr{O}(n-1)$ has n holomorphic sections. They transform as $\mathbf{n}$ under the action of $SU(2)$. The fixed point of left $U(1)$ action is the north and south poles of $\mathbb{CP}^2$, and the Weyl character formula states

$$e^{in\theta} + e^{i(n-1)\theta} + \cdots + e^{-in\theta} = \underbrace{\frac{e^{i(n+1/2)\theta}}{e^{i\theta/2} - e^{-i\theta/2}}}_{\text{from the north pole}} - \underbrace{\frac{e^{-i(n+1/2)\theta}}{e^{i\theta/2} - e^{-i\theta/2}}}_{\text{from the south pole}} . \tag{2.37}$$

2.5 Duistermaat-Heckmann theorem

Let us consider again $\mathrm{Ind}_{e^{i\beta J}}$. This time we take the rotation angle βJ proportional to the time β. The result is, by the fixed point theorem,

$$\mathrm{Tr}(-)^F e^{-\beta H} e^{i\beta J} = \sum_{\text{f.p.}} \prod_{\alpha=1}^{d/2} \frac{1}{e^{i\beta\theta_\alpha/2} - e^{-i\beta\theta_\alpha/2}}. \tag{2.38}$$

Hence, we take a non-trivial $\beta \to 0$ limit by multiplying the index by $(i\beta)^{d/2}$. The limit with J fixed gives

$$(i\beta)^{d/2}\mathrm{Ind}_{e^{i\beta J}} = \sum_{\text{f.p.}} \prod_{\alpha}^{d/2} \frac{1}{\theta_\alpha}. \tag{2.39}$$

In this section we show that this can be cast into the integrated form when M is symplectic and the action of g respects the symplectic structure:

$$\int e^{\omega+H} = \sum_{\text{f.p.}} \prod_{\alpha}^{d/2} \frac{1}{\theta_\alpha}. \tag{2.40}$$

where ω is the symplectic form and H is the Hamiltonian generating the vector field V for a. This is the celebrated theorem of Duistermaat and Heckmann. We will make use of this theorem afterwards in section 4.5.

Introduce first the equivariant differential D acting on differential forms on M:

$$Dx = dx - \iota_V x. \tag{2.41}$$

This operation is an anti-derivation with respect to the wedge product. A basic property is that $D^2 x = \mathscr{L}_V x$, i.e. it determines a complex on a V-invariant forms. A V-invariant form α with $D\alpha = 0$ is called equivariantly closed. A form of the form $D\alpha$ with α V-invariant is called equivariantly exact.

Note that the property that H generates V, i.e. $\iota_V\omega = dH$ translates into the condition $D(\omega + H) = 0$, i.e. $\omega + H$ is equivariantly closed. Another basic property is that

$$\int \alpha D\beta = -\int D\beta\alpha. \tag{2.42}$$

This means, among other things, that inclusion of equivariantly exact terms into the exponential in (2.40) does not alter the value:

$$\int e^{\omega+H+tD\alpha} = \int e^{\omega+H}, \tag{2.43}$$

because the derivative with respect to t of the right hand side becomes

$$\int (D\alpha) e^{\omega+H+tD\alpha} = -\int \alpha D e^{\omega+H+tD\alpha} = 0. \tag{2.44}$$

Let us now construct a V-invariant one form α. First, introduce on M a V-invariant metric g. (This is always possible if the group acting on it is compact. Just taking arbitrary metric and averaging with the group action do the job.) Then, define α by $\alpha(W) = g(V, W)$. This is clearly V-invariant. The equivariant differential of this is

$$D\alpha = -g(V, V) + d\alpha. \tag{2.45}$$

Using this $D\alpha$ in the equality (2.43) and taking the limit $t \to \infty$, the integration concentrates with the fixed point of the g action. The contribution of each fixed point is easily calculated by Gaussian integration and gives the desired expression (2.40). We recommend the reader the review [36] for more about the equivariant cohomology and related topics in physics.

Chapter 3

Seiberg-Witten theory

We review in this section the celebrated exact solution of $\mathcal{N} = 2$ super Yang-Mills theory initiated by Seiberg and Witten[8, 37]. It uses holomorphy inherent in supersymmetric theory and conjectural strong coupling duality, hence the argument involves some amount of hand-waving. However, the emerged view is highly consistent with each other and enabled us to reproduce various old conjectures on the strong coupling dynamics of field theory.

3.1 Generic structure of $\mathcal{N} = 2$ supersymmetric gauge theories

3.1.1 Prepotentials

Multiplets in the $\mathcal{N} = 2$ supersymmetry in four dimensions are of two types: one is the vector multiplet consisting of

$$\text{a vector } A_\mu, \quad \text{two Weyl fermion} \quad \lambda_1, \lambda_2, \quad \text{and a complex scalar } \phi. \tag{3.1}$$

In the language of $\mathcal{N} = 1$ supersymmetry, they comprise

$$\text{a vector multiplet } V = (\lambda_2, V_\mu) \tag{3.2}$$
$$\text{and a chiral multiplet } \Phi = (\phi, \lambda_1). \tag{3.3}$$

Another is the hypermultiplet which contains

$$\text{a Weyl fermion } \psi \text{ with representation } R \tag{3.4}$$
$$\text{two complex scalars } q, \tilde{q} \text{ with representation } R \oplus R^* \tag{3.5}$$
$$\text{a Weyl fermion } \tilde{\psi} \text{ with representation } R^*. \tag{3.6}$$

They are formed from

$$\text{a chiral multiplet in representation } R \;\; Q = (q, \psi) \tag{3.7}$$
$$\text{and a chiral multiplet in representation } R^* \;\; \tilde{Q} = (\tilde{q}, \tilde{\psi}). \tag{3.8}$$

Consider a system with n $U(1)$ vector multiplets V_i and n neutral chiral multiplets a_i. The generic Lagrangian which respects $\mathcal{N} = 1$ supersymmetry can be written as

$$\int d^4\theta K(a_i, a_i^\dagger) + \int d^4\theta \kappa_i V_i + \int d^2\theta (\frac{\tau_{ij}}{16\pi i}(a) W_i W_j + U(a_i)) + c.c. \tag{3.9}$$

Here

$$\tau_{ij} = \left(\frac{\theta}{2\pi} + i\frac{4\pi}{g^2}\right)_{ij} \tag{3.10}$$

is the complexified coupling constant. Expanding into components, the Lagrangian is

$$\begin{aligned}\mathscr{L} = &-|D_\mu a_i|^2 - g_{i\bar{j}} i\bar{\psi}^{\bar{j}}\sigma^\mu D_\mu \psi^i - \frac{1}{2}\left|\frac{\partial U}{\partial a_k} + \frac{\partial \tau_{ij}}{\partial a_k}\lambda_i^\alpha \lambda_{j\alpha}\right|^2 + \psi_i\psi_j\frac{\partial^2 U}{\partial a_i \partial a_j} \\ &+ \frac{\tau_{ij}}{16\pi i}\left(F_i^+ \wedge F_j^+ + \bar{\lambda}_i \sigma^\mu D_\mu \lambda_j\right) + \left(\frac{\tau_{ij}}{16\pi i}\right)^{-1}\left(\kappa_i + \frac{\partial \tau_{ik}}{\partial a_l}(\lambda_k\psi_l)\right)\left(\kappa_j + \frac{\partial \tau_{jm}}{\partial a_n}(\lambda_m\psi_n)\right) \\ &+ \frac{1}{2}(\lambda_i\lambda_j)(\psi_k\psi_l)\frac{\partial \tau_{ij}}{\partial a_k \partial a_l} + (\lambda_i^\alpha F_{j,\alpha\beta}\psi_k^\beta)\frac{\partial \tau_{ij}}{\partial a_k} + c.c.\end{aligned} \tag{3.11}$$

where we abbreviated $\psi^\alpha\psi_\alpha$ by $(\psi\psi)$, etc.

Let us determine the condition when this Lagrangian has another supersymmetry. The gauginos ψ and λ should appear symmetrically in the Lagrangian in order to respect the extended supersymmetry. This reduces U to be at most linear in a_i, that is $U = \zeta_i a_i$. Next,

$$\frac{\tau_{ij} - \tau_{ij}^\dagger}{8\pi i} = g_{i\bar{j}} = \frac{\partial^2 K}{\partial a_i \partial a_j^\dagger} \tag{3.12}$$

is required for the kinetic terms of ψ and λ to be equal. Regarding this as a differential equation for K, this can be solved if

$$K(a_i, a_j^\dagger) = \frac{a_D^i a_i^\dagger - a_D^{i\,\dagger} a_i}{8\pi i} \tag{3.13}$$

for some holomorphic functions a_D^i of a_i, and

$$\tau_{ij} = \frac{\partial a_D^i}{\partial a_i} = \frac{\partial a_D^j}{\partial a_i}. \tag{3.14}$$

This is integrable thanks to the symmetry of τ^{ij}, hence we have

$$a_D^i = \frac{\partial \mathscr{F}}{\partial a_i} \tag{3.15}$$

for some holomorphic function of a_i. With such K and τ^{ij}, the parameters $\mathrm{Re}\zeta$, $\mathrm{Im}\zeta$ and κ transform as a triplet under the $SU(2)$ R-symmetry. Let us recapitulate the result obtained so far. $\mathscr{N} = 2$ Lagrangian is controlled by a single holomorphic function $\mathscr{F}$ called prepotential, and

$$\mathscr{L} = \frac{1}{4\pi}\mathrm{Im}\left(\int d^4\theta \frac{\partial \mathscr{F}}{\partial a_i} a_i^\dagger + \int d^2\theta \frac{1}{2}\frac{\partial \mathscr{F}}{\partial a_i \partial a_j} W_i W_j\right) \tag{3.16}$$

in the absence of Fayet–Iliopoulos terms. a_i are called the special coordinates.

3.1.2 BPS multiplets and central charges

Let us recollect here another important implication that extended supersymmetry implies on the structure of the theory. The extended supersymmetry algebra satisfied by the supercharges is

$$\{Q^i_\alpha, (Q^j_\beta)^\dagger\} = P_{\alpha\dot{\beta}}\delta^{ij}, \tag{3.17}$$

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij} \tag{3.18}$$

where Z^{ij} is a complex antisymmetric matrix, $i = 1, \ldots, N$. Notice that the $\{Q^i_\alpha, Q^j_\beta\}$ terms are in general nonzero. They are shown to commute with every other generator in the supersymmetry algebra, hence the name central charges.

Let us study the representation of supersymmetry in the presence of central charges. Choose the coordinate frame so that $P^\mu = (M, 0, 0, 0)$ and make

$$Z_{ij} = \begin{pmatrix} 0 & Z_1 & & & \\ -Z_1 & 0 & & & \\ & & 0 & Z_2 & \\ & & -Z_2 & 0 & \\ & & & & \ddots \end{pmatrix} \tag{3.19}$$

by an appropriate $U(N)$ rotation. Then it is easy to see that Q^i_α can be reorganized into N pairs of fermionic oscillators with

$$\{a_i, a_i^\dagger\} = M + |Z_i|, \qquad \{b_i, b_i^\dagger\} = M - |Z_i|, \tag{3.20}$$

Hence the representation of the supersymmetry algebra drastically changes in the presence of non-zero central charges. The number of states in a multiplet can be reduced in a power of two with respect to the ordinary massive multiplet with 2^{2N} states when one of the central charges equals the mass. These states with reduced number of components are called Bogomolny-Prasad-Sommerfield states, or BPS states for short. The reduced number of states indicates a certain sense of stability in these BPS states, since a BPS state needs another BPS state in order to make a non-BPS state which has twice the number of components. Hence generically the mass is locked proportional to the central charge and cannot receive corrections.

Finally, restricting the discussion to the $\mathcal{N} = 2$ supersymmetry, let us see the relationship between the central charges and the special coordinates. For example, consider a hypermultiplet with charge n is coupled to the system of the vector multiplets described by the Lagrangian (3.16). The relevant part of the action for the hypermultiplet is, in $\mathcal{N} = 1$ notation,

$$\mathcal{L}_{\text{hyper}} = \int d^4\theta (Q^\dagger e^V Q + \tilde{Q}^\dagger e^{-V} \tilde{Q}) + \int d^2\theta (\sqrt{2} Q a \tilde{Q} + M Q \tilde{Q}) + c.c \tag{3.21}$$

The superpotential term is fixed in this form, because this is related by the extended supersymmetry to the minimal coupling of the multiplet to the gauge field. This reveals that their mass is nonzero and equal to

$$\sqrt{2} n |a + M| \tag{3.22}$$

Since the hypermultiplet consists of only four bosonic degree of freedom, it must be a BPS multiplet in order to be massive. This suggests the identification of the central charges and the special coordinates, although equation (3.22) alone does not fix the relative phase between the two. When we further consider a hypermultiplet with charge (m, n) with respect of two $U(1)$ factors with its special coordinate a and a', a similar calculation shows that their mass is equal to $\sqrt{2}|ma + na' + M|$. This shows that there should be no relative phase between the special coordinates and the central charges. From now on, we use these two words interchangeably.

3.1.3 Electromagnetic duality

The Maxwell equation, without the presence of source terms, has a symmetry exchanging the electric field and the magnetic field. To phrase this in a relativistically invariant way, the symmetry exchanges

$$F_{\mu\nu} \longleftrightarrow *F_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{3.23}$$

Since this naïve form of electromagnetic duality is difficult to formulate in the presence of charged matter, and since coupling constants of $U(1)$ gauge fields without any charged matter are apparently meaningless, it may seem vacuous to discuss the change in the coupling constant of the gauge theory under the duality. Witten showed [38], however, that quantum free $U(1)$ gauge theories have a dependence of its coupling constant once they are formulated on closed compact manifolds of dimension four, and that the electromagnetic duality acts on the partition function as the modular transformation. Let us first very briefly review this electromagnetic duality.

duality for abelian gauge fields

Consider n $U(1)$ gauge fields A_i with complexified coupling constants τ_{ij}. Put the system on a closed spin manifold M of dimension four. The action is

$$S = \int \frac{1}{g^2}F_i \wedge *F_j + \int i\frac{\theta_{ij}}{8\pi^2}F_i \wedge F_j \tag{3.24}$$

$$= \int \frac{i\bar{\tau}}{4\pi}F^+ \wedge F^+ + \int \frac{i\tau}{4\pi}F^- \wedge F^-, \tag{3.25}$$

where we defined $F^\pm = (F \pm *F)/2$. Note that the θ term is normalized to have symmetry

$$\theta_{ij} \sim \theta_{ij} + 2\pi n_{ij} \tag{3.26}$$

where n_{ij} is a matrix with integral entries, since $F_i/(2\pi)$ defines an integral cohomological class and the intersection form of a spin manifold is even.

Consider extending the action by introducing gauge fields C_i and two-form fields G_i to

$$S' = \int \frac{i\bar{\tau}}{4\pi}\mathcal{F}^+ \wedge \mathcal{F}^+ + \int \frac{i\tau}{4\pi}\mathcal{F}^- \wedge \mathcal{F}^- - \frac{i}{2\pi}\int F_C \wedge G \tag{3.27}$$

with $\mathcal{F} = F_A - G$. This system has a gauge invariance with

$$A \to A + B, \qquad G \to G + F_B \tag{3.28}$$

where B is the curvature of some line bundle. The addition of two connection should be interpreted as the tensoring of line bundles. The gauge invariance of the last term is guaranteed by $[F_C] \in H^2(M, \mathbb{Z})$.

Firstly, this Lagrangian is equivalent to S. It is because we obtain $dG = 0$ and $[G] \in H^2(M, \mathbb{Z})$ when C is integrated out. This means G can be gauged away using (3.28). On the other hand, we can first gauge away A using the gauge invariance (3.28). This gives the Lagrangian

$$\int \frac{i\bar{\tau}}{4\pi} G^+ \wedge G^+ + \int \frac{i\tau}{4\pi} G^- \wedge G^- - \frac{i}{2\pi} \int F_C \wedge G. \tag{3.29}$$

The action for G is now Gaussian, and the result of integration is

$$S = -\int \frac{i}{4\pi\bar{\tau}} F_C^+ \wedge F_C^+ - \int \frac{i}{4\pi\tau} F_C^- \wedge F_C^-, \tag{3.30}$$

This is just the original Lagrangian (3.25) with τ replaced by $-1/\tau$. Combined with transformation (3.26), these generate the symmetry acting on τ by

$$\tau' = (A\tau + B)(C\tau + D)^{-1} \qquad \text{where} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{Z}). \tag{3.31}$$

There is another way to understand the appearance of the symplectic group. In general, a particle is labeled by its electric and magnetic charges (e^i, m_i). Between two such charge vectors (e^i, m_i) and (e'^i, m'_i), there is a natural symplectic pairing

$$e^i m'_i - m_i e'^i. \tag{3.32}$$

This measures the phase acquired by circulating a particle with charge (e^i, m_i) once around a particle with charge (e'^i, m'_i). The group $Sp(2n, \mathbb{Z})$ can be thought of as acting on the charge vectors (e^i, m_i) respecting this symplectic pairing.

action of duality on the scalar fields

Pure $\mathcal{N} = 2$ $U(1)^n$ gauge theory is almost free, except the dependence of the coupling constant to the scalar field. Inspecting the derivation of the electromagnetic duality above, we see that we can dualize the gauge field in this $\mathcal{N} = 2$ supersymmetric setting. We saw in section 3.1.1 that the coupling constant satisfies the relation $\tau_{ij} = \partial a_D^i / \partial a_j$. Since the duality changes τ_{ij}, a_i should be transformed accordingly. This is achieved if we introduce new coordinates a' and a'_D by

$$\begin{pmatrix} a_D^i{}' \\ a'_i \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_D^i \\ a_i \end{pmatrix}. \tag{3.33}$$

The new prepotential must be recalculated from the relation

$$a_D^i{}' = \frac{\partial \mathcal{F}'}{\partial a'_i}. \tag{3.34}$$

These relation suggests the variables a_D^i as the electromagnetic dual of the special coordinates a_i. From the consideration in the previous section, these a_D^i should control the masses of BPS states charged under the dualized $U(1)$, or stated more plainly, the BPS magnetic monopoles of the original $U(1)$. Let us see this explicitly in a classical calculation. The energy of a static configuration is, from the Lagrangian (3.16),

$$E = \frac{1}{4\pi}\int d^3x D_\mu a_i (\mathrm{Im}\tau)^{ij} D_\mu \bar{a}_j + \frac{1}{2} B_i^\mu \tau^{ij} (\mathrm{Im}\tau)^{-1}_{jk} \bar{\tau}^{kl} B_l^\mu. \tag{3.35}$$

This expression can be reorganized as

$$= \frac{1}{4\pi}\int d^3x D_\mu a_D^i \left(\mathrm{Im}\frac{-1}{\tau}\right)_{ij} D_\mu \bar{a}_D^j + \frac{1}{2} B_i^\mu \left(\mathrm{Im}\frac{-1}{\tau}\right)^{-1\ ij} B_j^\mu \tag{3.36}$$

$$\geq \frac{\sqrt{2}}{4\pi}\left|\int dx^3 B_i^\mu D_\mu a_D^i\right| = \frac{\sqrt{2}}{4\pi}\left|\oint_S B_i^\mu a_D^i dn_\mu\right| = \sqrt{2}|m_i a_D^i|. \tag{3.37}$$

This shows that the BPS central charge for a monopole is indeed a_D.

3.2 Seiberg-Witten solution

At generic points in the moduli, pure $\mathcal{N} = 2$ $SU(N)$ super Yang-Mills is Higgsed down to $U(1)^{N-1}$ by the vacuum expectation value of the adjoint scalar. In 1994, Seiberg and Witten determined the low-energy prepotential governing this $U(1)^{N-1}$ gauge theory, by utilizing the holomorphy of the prepotential and some physical assumptions on the electro-magnetic duality. We review in this section this achievement.

3.2.1 Pure $SU(2)$

Classical analysis

For pure $SU(2)$ gauge theories, the potential for the adjoint scalar ϕ contains a term

$$\propto [\phi, \phi^\dagger]^2.$$

This means that ϕ can be diagonalized at vacuum. Let us write

$$\phi = \mathrm{diag}(a, -a).$$

a and $-a$ should be identified because they are related by a Weyl reflection. When a is nonzero, the gauge group is broken down to $U(1)$ commuting with ϕ. The prepotential describing the dynamics of this $U(1)$ is

$$F = \frac{1}{2}\tau_0 a^2,$$

where τ is the complexified gauge coupling.

Perturbative one-loop analysis

In quantum theory, the gauge coupling runs according to the renormalization group equation. At one loop, they are easily determined by the representation content of various fields and in this case

$$\tau_{\Lambda_1} = \tau_{\Lambda_0} + \frac{2i}{\pi}\log\Lambda_1/\Lambda_0 \tag{3.38}$$

From this we see the combination

$$\Lambda^4 = \Lambda_0^4 e^{2\pi i \tau_{\Lambda_0}}$$

is renormalization-group invariant. The coupling constant of the unbroken $U(1)$ is obtained by setting $\lambda_0 = a$ in the above formula, because the running stops when the momentum scale drops below the mass of charged W bosons $\sim a$. To reproduce this coupling, the one-loop corrected prepotential should be

$$\mathscr{F} = \frac{i}{2\pi} a^2 \log \frac{a}{\Lambda} \tag{3.39}$$

The modification of the prepotential from the running of the coupling has another physical manifestations. Calculating dual special coordinates from the above formula we obtain

$$a_D = \frac{\partial \mathscr{F}}{\partial a} = \frac{2ia}{\pi} \log \frac{a}{\Lambda} + \frac{ia}{\pi} \tag{3.40}$$

This signifies, when the phase of a is changed adiabatically from 0 to π, the resulting a_D becomes $-a_D + 2a$ in the original variables. The effect is in full accord with the finding by Witten [39], where he noticed that adiabatically changing the theta angle, a magnetic monopole becomes a dyon. All these findings can be succinctly summarized by saying that, when $u = \mathrm{tr}\phi^2$ loops around ∞, there is a monodromy

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{3.41}$$

acting on (a_D, a).

Analysis of the non-perturbative region

Classically, full $SU(2)$ gauge symmetry is restored when $a = 0$. This is reflected in the one-loop analysis in that there is a logarithmic singularity $\mathscr{F} \sim a^2 \log a^2$ in the prepotential. There is however a serious problem in extrapolating the one-loop prepotential to the deep non-perturbative region. The problem is that, the gauge coupling for the unbroken $U(1)$, becomes negative for sufficiently small a. It can be rephrased as follows: the inverse of the coupling constant $1/g^2$ should be a harmonic function of a from supersymmetry. However, a harmonic function, defined on all of a, cannot have a minimum and hence it must be negative somewhere. In order to escape this argument, we should allow the coupling to be multivalued.

Here comes the electromagnetic duality to the rescue. $U(1)$ gauge theories have the duality which exchanges electric field $F_{\mu\nu}$ and its magnetic dual $*F_{\mu\nu}$. This operation reverses the coupling constant:

$$S : \tau \to -\frac{1}{\tau}. \tag{3.42}$$

There is another transformation which does not change the physics:

$$T : \tau \to \tau + 1. \tag{3.43}$$

These two transformations generate $SL(2,\mathbb{Z})$ symmetry group. The argument in the previous paragraph indicates that, in order to have a positive coupling constant everywhere, there should be some point in the moduli around which we need S transformation.

Let us determine how many of these singularities are necessary. First, let us count the number of global symmetries acting on the moduli space. Taking the gauge invariant polynomial $u = \langle \mathrm{tr}\phi^2 \rangle$ rather than the eigenvalue a as the parameter for the modulus, we see that the Weyl reflection $a \to -a$ acts trivially on u. However, there is another unbroken global symmetry taking $u \to -u$. This is because, as $\Lambda \sim e^{i\theta/4}$, changing $\theta \to \theta + 2\pi$ sends $\Lambda^2 \to -\Lambda^2$. All physical quantities are determined by u/Λ^2, this translates to the action $u \to -u$. Hence, a minimum number of singularities are two, one at $u = t\Lambda^2$ and another at $u = -t\Lambda^2$. We take the renormalization prescription such that $t = 1$.

Let us next compute the monodromy of $SL(2,\mathbb{Z})$ around these singularity. Denote as $M_\pm$ the monodromy around $u = \pm\Lambda^2$, respectively. These should satisfy the constraint

$$M_\infty = M_- M_+. \tag{3.44}$$

Furthermore, as the symmetry $u \to -u$ is generated by $\theta \to \theta + 2\pi$, we have

$$M_- = T M_+ T^{-1}. \tag{3.45}$$

These two condition determine the matrices. The results are

$$M_+ = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad M_- = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}. \tag{3.46}$$

We now have enough information to determine a and a_D, since the three mondromies determine a holomorphic bundle of which a and a_D is the sections, and since we know the asymptotic behavior of a and a_D around infinity

$$a \sim \sqrt{2u} \qquad a_D \sim i\sqrt{2u}\pi \log u. \tag{3.47}$$

Let us examine the singularity in more detail. By taking a duality transformation exchanging a and a_D, M_+ becomes

$$S M_+ S^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{3.48}$$

This suggests an interpretation that there is one hypermultiplet charged with respect to the dualized gauge field that becomes massless at $u = \Lambda^2$, and that the monodromies are caused by the one-loop effect of this hypermultiplet. Indeed, as $a_D = 0$ and $a \neq 0$ at $u = \Lambda^2$, a magnetic monopole with respect to the original $U(1)$ is becoming massless at that point.

Phrasing the result in terms of curves

The result found above can be summarized beautifully in a geometrical way. Consider a family of curves X_u parametrized by u:

$$y^2 = (x^2 - u)^2 - \Lambda^4 \tag{3.49}$$

and a meromorphic differential

$$d\lambda = \frac{2x^2 dx}{y} \tag{3.50}$$

on the curve. Take two contour A and B as in figure 3.1. Consider following quantities

$$\alpha = \frac{1}{2\pi i}\oint_A d\lambda, \qquad \alpha_D = \frac{1}{2\pi i}\oint_B d\lambda. \tag{3.51}$$

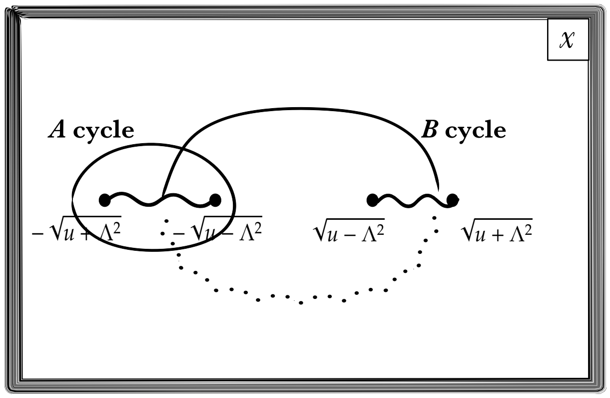


Figure 3.1: Elliptic curve and the contour A, B

We can check that the contours A and B have the same monodromy $M_\pm$ and M_∞ when u loops respectively around $\pm\Lambda$ and ∞. Hence, α and α_D also transforms accordingly. Furthermore, we can check in the weak coupling regime $\Lambda \ll u$ that

$$\alpha \sim \sqrt{2u} \qquad \alpha_D \sim i\sqrt{2u}\pi \log u. \tag{3.52}$$

These properties shows that α and α_D are none other than the special coordinates a and a_D.

Differentiating these expressions by u, we obtain

$$\frac{\partial a}{\partial u} = \frac{1}{2\pi i}\oint_A \frac{dx}{y}, \qquad \frac{\partial a}{\partial u} = \frac{1}{2\pi i}\oint_B \frac{dx}{y} \tag{3.53}$$

Notice the integrand is now a holomorphic differential, that is, it has no poles. For a point p in the curve X_u, consider a quantity

$$f : p \to \int_{p_0}^{p} \omega \tag{3.54}$$

using a holomorphic differential ω. As the path from p_0 to p is determined only up to the addition of a multiple of the contour A and B, the map f is defined only up to the addition of $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$. Hence f determines a map to $\mathbb{C}/\Lambda$ where Λ is the lattice with bases $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$. This is another standard representation for an elliptic curve, and we see the complexified gauge coupling

$$\tau = \frac{\partial a_D}{\partial a} = \frac{\partial a_D/\partial u}{\partial a/\partial u} \tag{3.55}$$

is none other than the complex structure of the elliptic curve. This guarantees that $\mathrm{Im}\tau$ is positive, as it should be as the squared coupling constant of gauge theory.

3.2.2 Extension to the gauge group $SU(N)$

Extension of the above considerations to the $SU(N)$ gauge groups is carried out by Argyres and Faraggi[40] and by Klemm *et al.*[41]. At low energy at generic points in the moduli

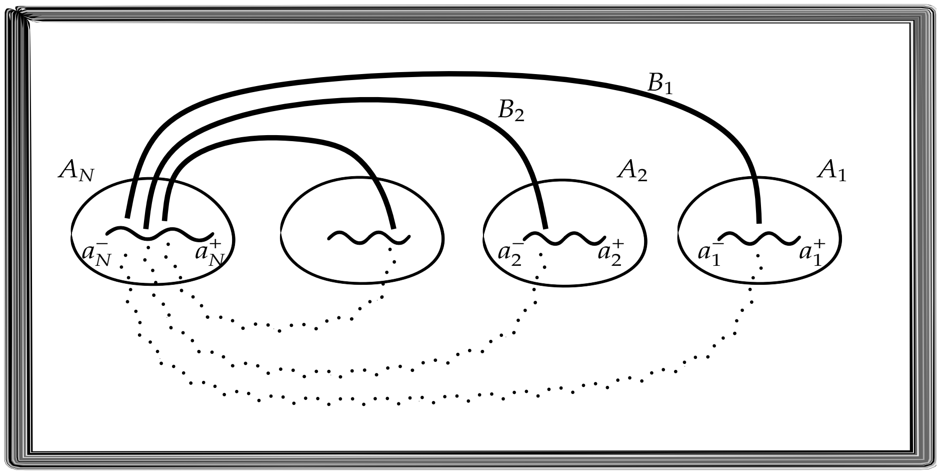


Figure 3.2: the curves for $SU(N)$

space, the vacuum expectation value of the adjoint scalar breaks the gauge groups to $U(1)^{N-1}$. The complexified coupling constants τ_{ij} should satisfy the constraint that $\mathrm{Im}\tau_{ij}$ is positive definite. We saw that there is an action of electromagnetic duality $Sp(2n,\mathbb{Z})$ acting on the central charges a_i and a_D^i.

All these structures suggest that there is some genus $N-1$ Riemann surface governing the dynamics of the $N-1$ $U(1)$ vector multiplets, so that the charge vectors are related to the integral homology of the surface and the coupling constants τ_{ij} comprise the period matrix of the surface. In view of the fact that the curve for the $SU(2)$ gauge theory is

$$y^2 = (x^2-u)^2 - \Lambda^4$$

a natural guess for the curve is a hyperelliptic

$$y^2 = P(x)^2 - \Lambda^{2N} \tag{3.56}$$

where

$$P(x) = \langle \det(x-\phi)\rangle = \sum u_p x^{n-p} \tag{3.57}$$

. It is also natural to suggest the form of the special coordinates:

$$a_k = \frac{1}{2\pi i}\oint_{A_k} d\lambda, \qquad a_D^k = \frac{1}{2\pi i}\oint_{B^k} d\lambda \tag{3.58}$$

where $d\lambda$ is some meromorphic differential with no residue. Here the placement of the contours A_k and B^k are depicted in figure 3.2. We can show that by choosing

$$d\lambda = \frac{xdP(x)}{y} \tag{3.59}$$

one can obtain satisfying results.

Firstly, consider the derivative of a_k and a_D^k with respect to the classical moduli u_p. They are given by

$$\frac{\partial a_k}{\partial u_p} = \frac{1}{2\pi i}\oint_{A_k} \frac{\partial}{\partial u_p} d\lambda, \qquad \frac{\partial a_D^k}{\partial u_p} = \frac{1}{2\pi i}\oint_{B^k} \frac{\partial}{\partial u_p} d\lambda. \tag{3.60}$$

An important property of our choice (3.59) is that the derivatives of the differential $d\lambda$ are all holomorphic on the hyperelliptic including the point at infinity. Hence they form the basis of holomorphic one forms on the curve. As such, the quantity

$$\tau^{ij} = \frac{\partial a_D^i}{\partial a_j} = \sum_p \frac{\partial a_D^i}{\partial u_p}\frac{\partial u_p}{\partial a_j} = \sum_p \frac{\partial a_D^i}{\partial u_p}\left(\frac{\partial a_j}{\partial u_p}\right)^{-1} \tag{3.61}$$

is precisely the period matrix of the curve. The period matrix has two important property[42]. One is that it is symmetric,

$$\tau_{ij} = \tau_{ji} \tag{3.62}$$

and another is that the imaginary part is positive definite,

$$\mathrm{Im}\tau_{ij} > 0. \tag{3.63}$$

The first condition shows that the dual special coordinates a_D^k are integrable and are obtained as the derivatives of some function, that is

$$a_D^k = \frac{\partial \mathscr{F}}{\partial a_k}. \tag{3.64}$$

This is as expected since the system has $\mathcal{N} = 2$ supersymmetry and hence can be described by a prepotential. The second condition, on the other hand, guarantees that the gauge theory has positive squared gauge couplings. This is necessary for the stability of the system. We will see in the next section that the prepotential obtained from the curve has the weak-coupling expansion of the form just as expected from physical consideration.

3.3 Expansion of the exact prepotential

In the previous section, we obtained a candidate exact prepotential from the consideration of various strong coupling limits with the help of electro-magnetic duality. The result is, however, expressed in a very indirect terms using line integrals of a certain meromorphic differential form over a complex curve. To rephrase the answer in the form that is suitable for comparison to the weak coupling results, we first need to solve the BPS central charges a_d for magnetic monopoles in terms of a for electrically charged particles. It is not an easy task and even the verification of the presence of the logarithmic one-loop contribution in the Seiberg-Witten prepotential necessitates some efforts. As we want to compare the results from the strong coupling dualities against a straightforward weak-coupling instanton calculation, we should now attack the problem.

There have been two ways to overcome this, one is the use of Picard-Fuchs equation which the periods a and a_D satisfy. The method is very well suited for gauge groups with lower rank, but it becomes cumbersome very quickly. In this section, we review another method developed by D'Hoker and his collaborators.

3.3.1 Classical and quantum moduli

Firstly, let us see how we can express the central charges a_i for electric $U(1)$ by means of classical moduli $\tilde{a}_i$ appearing in the definition of the curves,

$$P(x) = \prod (x - \tilde{a}_i). \tag{3.65}$$

We follow the exposition by D'Hoker, Krichever and Phong[43]. For the sake of conciseness, we limit the derivation to the pure $SU(N)$ case. All the argument can be readily extended to theories with additional hypermultiplets.

Let us expand a_k in powers of Λ. To do this, first we fix the contour A_k independent of Λ. Then we can expand the denominator y in the integrand as

$$2\pi i a_k = \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \oint_{A_k} dx \frac{xP'}{P} \left(\frac{\Lambda^{2N}}{P^2}\right)^m . \tag{3.66}$$

Rewriting the integrand using

$$\frac{xP'}{P}\left(\frac{\Lambda^{2N}}{P^2}\right)^m = -\frac{d}{dx}\left(\frac{x}{2m}\left(\frac{\Lambda^{2N}}{P^2}\right)^m\right) + \frac{1}{2m}\left(\frac{\Lambda^{2N}}{P^2}\right)^m , \tag{3.67}$$

the residue can be easily computed. The result is

$$a_k = \tilde{a}_k + \sum_{m=1}^{\infty} \frac{\Lambda^{2m}}{2^{2m}(m!)^2} \left(\frac{\partial}{\partial \tilde{a}_k}\right)^{2m-1} S_k(\tilde{a}_k)^m, \tag{3.68}$$

where

$$S_k(x) = \frac{\Lambda^{2(N-1)}}{\prod_{i\neq k}(x - \tilde{a}_i)^2}. \tag{3.69}$$

3.3.2 Verification of the one-loop terms

Prepotential, in its weak coupling region, will have the form

$$\mathscr{F} = \frac{N}{\pi i}\sum a_k^2 + \frac{i}{4\pi}\sum_{i<j}(a_i - a_j)^2 \log\frac{(a_i-a_j)^2}{\Lambda^2} + \sum_{m=1}^{\infty}\frac{\Lambda^{2Nm}}{2m\pi i}\mathscr{F}^{(m)}(a) \tag{3.70}$$

with the first and second term being the classical and one-loop contribution, and the rest the instanton correction. There should be no higher order loop correction. This follows from consideration based on holomorphy and the shift symmetry for θ. Indeed, n-loop terms should be accompanied by the factor g^{2n-2}. Holomorphy shows that they should be augmented to the combination $\tau^{(1-n)}$. Perturbation theory should have the symmetry under the continuous shift of $\theta = \mathrm{Re}\tau$. Hence perturbative correction is possible only for $n = 1$. The power of Λ in the expansion above is also determined by the holomorphic structure and the anomaly. Firstly from the renormalization group equation, the dynamically generated scale is proportional to

$$\Lambda = \Lambda_0 e^{\frac{\pi}{N} i \tau_{\Lambda_0}}. \tag{3.71}$$

Hence, k-instanton contribution, which should be accompanied by the factor $e^{ik\theta}$, appears with the factor Λ^{2Nk}.

Let us see in this subsection that the prepotential calculated from the Seiberg-Witten curves are of the form expected from this weak coupling analysis. In order to check the behavior of $\mathscr{F}$, It suffices to directly evaluate the dual special coordinate a_D^i in a weak coupling regime using the line integral representation.

Take the contour B^i as indicated in the figure 3.2. In the weak coupling limit, the cuts are situated at $\tilde{a}_i^- \sim \tilde{a}_i^+$ both very near $\tilde{a}_i$. The contour B^i goes from $\tilde{a}_1^-$ to $\tilde{a}_i^-$ in the upper patch, then passes through the cut and goes back to $\tilde{a}_1^-$. It is important to note that $\tilde{a}_i^\pm$ can be expanded into powers of Λ^{2N} with coefficients rational in a_k.

The term to calculate is

$$2\pi i \int_{B^i} d\lambda = \int_{B^i} \frac{xP'(x)dx}{\sqrt{P(x)^2 - \lambda^{2N}}} \tag{3.72}$$

$$= 2\int_{x_1^-}^{x_i^-} \frac{xP'(x)dx}{\sqrt{P(x)^2 - \lambda^{2N}}}. \tag{3.73}$$

As the end points of integration themselves depend on Λ, naïve expansion of integrand leads to apparent non convergence. To circumvent this, we introduce an artificial parameter ξ and consider

$$2\pi i a_D^k(\xi) = 2\int_{x_1^-}^{x_i^-} \frac{xP'(x)dx}{\sqrt{P(x)^2 - \xi^2\lambda^{2N}}} \tag{3.74}$$

$$= 2\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)}\xi^{2m} \int_{x_1^-}^{x_i^-} dx \frac{xP'}{P}\left(\frac{\Lambda^{2N}}{P^2}\right)^m. \tag{3.75}$$

We take analytic continuation $\xi \to 1$ afterwards. The integrand can be rewritten as

$$\frac{xP'(x)}{\sqrt{P(x)^2 - \xi^2\Lambda^{2N}}} = \sum_l \frac{1}{x-\tilde{a}_l}\frac{1}{2\pi i}\oint_{A_l} dz \frac{zP'(z)}{\sqrt{P(z)^2-\xi^2\Lambda^{2N}}} + N + \sum\sum_{p\geq 2}\frac{\mathcal{O}(\Lambda^{2N})}{(x-\tilde{a}_l)^p}. \tag{3.76}$$

First carrying out the integral and then taking $\xi \to 1$, we obtain

$$2\pi i a_D^k = 2\sum a_l \log(\tilde{a}_k^- - \tilde{a}_l) + 2N_c\tilde{a}_k^- - 2\sum\sum_{p\geq 2}\frac{1}{p-1}\frac{\mathcal{O}(\Lambda^{2N})}{(x-\tilde{a}_l)^{p-1}}. \tag{3.77}$$

Since

$$\tilde{a}_k^- = \tilde{a}_k + \Lambda^{2N} + \mathcal{O}(\Lambda^{4N}) \tag{3.78}$$

to first order,we obtain

$$2\pi i a_D^k = 2Na_k \log\Lambda - \sum_{l\neq k}(a_k - a_l)\log(a_k - a_l)^2 + \mathcal{O}(\Lambda^{2N}). \tag{3.79}$$

Integrating this with respect to $a_k - a_1$, we find that the prepotential determined from the Seiberg-Witten curve indeed has the form (3.70).

3.3.3 Renormalization group equation

We show in this subsection the renormalization group equation satisfied by the prepotential:

$$\Lambda\frac{\partial\mathscr{F}}{\partial\Lambda}=\frac{N}{\pi i}\sum\tilde{a}_k^2. \tag{3.80}$$

The relation for pure $SU(2)$ theory was originally obtained by Matone[44]. General relation (3.80) was independently obtained by Eguchi and Yang[45] and by Sonnenschein, Theisen and Yankielowicz[46].

Let us begin the derivation. First, since the prepotential is of dimension two, it satisfies the Euler equation

$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\sum a_k\frac{\partial}{\partial a_k}\right)\mathscr{F}=2\mathscr{F}. \tag{3.81}$$

Differentiating this by the symmetric polynomials u_p, we obtain

$$\frac{\partial}{\partial u_p}\Lambda\frac{\partial}{\partial\Lambda}\mathscr{F}=\sum\left(a_k\frac{\partial a_D^k}{\partial u_p}-\frac{\partial a_k}{\partial u_p}a_D^k\right) \tag{3.82}$$

$$=\frac{1}{(2\pi i)^2}\sum\left(\oint_{A_k}d\lambda\oint_{B^k}\frac{\partial}{\partial u_p}d\lambda-\oint_{A_k}\frac{\partial}{\partial u_p}d\lambda\oint_{B^k}d\lambda\right). \tag{3.83}$$

By using the Riemann bilinear relation[42], this can be further simplified to

$$=\frac{1}{2\pi i}\sum_{p\text{: poles of }\lambda}(\text{residue at }p)\int_{p_0}^{p}\frac{\partial}{\partial u_p}d\lambda. \tag{3.84}$$

Thus, we obtained the relation

$$2\pi i\frac{\partial}{\partial u_p}\Lambda\frac{\partial}{\partial\Lambda}\mathscr{F}=\begin{cases}2N & \text{when } p=2,\\ 0 & \text{otherwise.}\end{cases} \tag{3.85}$$

Integration of this relation shows the renormalization group equation (3.80). Here, the integration constant is zero due to the homogeneity of the prepotential.

3.3.4 Linear recursion relation

Following Chan and D'Hoker[47], we derive a linear recursion relation determining the instanton correction in the prepotential determined from the curve. It is obtained by combining the renormalization group equation derived in the previous section and the weak-coupling ansatz confirmed in section 3.3.2. There are other recursion relations that the prepotential obtained from the curve satisfy, most notably the Witten-Dijkgraaf-Verlinde-Verlinde formula (for example, see [48, 49]). Even though they may have more mathematical relevance, the recursion relation so obtained are quadratic and in reality they are rather cumbersome as the method to extract the instanton correction.

Firstly, let us calculate $(\Lambda\partial/\partial\Lambda)\mathscr{F}$ by substituting the weak coupling ansatz equation (3.70). Putting this into the renormalization group equation (3.80), we get

$$\sum\tilde{a}_k^2=\sum a_k^2+\sum_{m=1}^{\infty}\Lambda^{2Nm}\mathscr{F}^m(a). \tag{3.86}$$

A further rewriting the quantum moduli a_k by the classical moduli $\tilde{a}_k$ using equation (3.68) yields

$$0 = \sum_k \left(\sum_{m=0}^{\infty} \Lambda^{2Nm} \Delta_k^{(m)}(\tilde{a}) \right)^2 - \sum_k \left(\Delta_k^{(0)}(\tilde{a}) \right)^2 + \sum_{m=1}^{\infty} \Lambda^{2Nm} \mathscr{F}^{(m)} \left(\sum_{n=0}^{\infty} \Lambda^{2Nn} \Delta_k^{(n)}(\tilde{a}) \right), \tag{3.87}$$

where we defined for brevity $\Delta_k^{(0)}(x) = \tilde{a}_k$ and

$$\Delta_k^{(m)}(x) = \frac{1}{2^{2m}(m!)^2} \left(\frac{\partial}{\partial x} \right)^{2m-1} S_k(x)^m. \tag{3.88}$$

Since the equation above is expressed in terms of $\tilde{a}$ alone, we can replace them by a. By expanding in powers of Λ^2 and comparing the term of order Λ^{2Nm}, we now have a recursion relation defining the instanton coefficient $\mathscr{F}^{(m)}$ using the same instanton coefficient $\mathscr{F}^{(n)}$ with lower degree, that is, $n < m$. Now the functions $\Delta_k^{(m)}$ also should be redefined by substituting $\tilde{a}$ by a.

We collect here some of the low instanton results:

$$-\mathscr{F}^{(1)} = \sum 2\Delta_k^{(0)} \Delta_k^{(1)}, \tag{3.89}$$

$$-\mathscr{F}^{(2)} = \sum (2\Delta_k^{(0)} \Delta_k^{(2)} + (\Delta_k^{(1)})^2) + \sum \Delta_k^{(1)} \frac{\partial}{\partial a_k} \mathscr{F}^{(1)}, \tag{3.90}$$

$$\begin{aligned} -\mathscr{F}^{(3)} = &\sum (2\Delta_k^{(0)} \Delta_k^{(3)} + 2\Delta_k^{(1)} \Delta_k^{(2)}) \\ &+ \sum \left(\Delta_k^{(1)} \frac{\partial}{\partial a_k} \mathscr{F}^{(2)} + \Delta_k^{(2)} \frac{\partial}{\partial a_k} \mathscr{F}^{(1)} \right) + \frac{1}{2} \sum \Delta_k^{(1)} \Delta_l^{(1)} \frac{\partial^2}{\partial a_k \partial a_m} \mathscr{F}^{(1)}. \end{aligned} \tag{3.91}$$

This is matched against direct instanton calculation in the next chapter.

Chapter 4

Instanton calculation of the prepotential

We reviewed in the last chapter the works initiated by Seiberg and Witten. Holomorphy inherent in supersymmetric theories and a few educated guesses enabled us to determine the effective prepotential which governs the low energy properties of $\mathcal{N} = 2$ $SU(N)$ gauge theories. Furthermore, we saw in section 3.3.2 that the result so obtained can be expanded in the weak coupling region in the form

$$\mathscr{F} = \underbrace{\frac{N}{\pi i}\sum a_k^2 + \frac{i}{4\pi}\sum_{i<j}(a_i - a_j)^2 \log\frac{(a_i - a_j)^2}{\Lambda^2}}_{\text{classical + one-loop}} + \underbrace{\sum_{m=1}^{\infty}\frac{\Lambda^{2Nm}}{2m\pi i}\mathscr{F}^{(m)}(a)}_{\text{instanton effects}}. \tag{4.1}$$

This expression admits a natural interpretation as a contribution of one-loop term plus instanton effects.

Since we know the moduli space of the multi-instanton thanks to the work of Atiyah, Drinfeld, Hitchin and Manin[5], we can, in principle, carry out an semi-classical calculation order-by-order around the instanton solution and compare the result against the coefficients $\mathscr{F}^{(m)}(a)$ in (4.1). Soon after the work of Seiberg and Witten, several groups initiated the study along that direction. It is, however, rather difficult to carry out the instanton calculation at high instanton number or higher rank gauge groups. One of the reason behind the difficulty is the complexity of the topology of the moduli space.

Another important subtlety to mention is that, in the presence of the non-zero vacuum expectation values for the adjoint scalar fields, an anti-self-dual connection is no longer a solution of the equation of motion. The proper way of treatment was clarified by Affleck[50] and the method was called the constrained instanton method. The end result is that a non-constant potential, called the constrained instanton action, is induced on the instanton moduli at the leading order in the coupling constant. This reflects the fact that they no longer satisfies the equation of motion. Higher order corrections can be computed order-by-order. However when the quantity we want to calculate is protected by holomorphy, we can often make an argument that guarantees that no higher order correction in the gauge coupling constant is possible. In these cases, lowest order calculation using the constrained instanton action should give the exact answer.

The hard work of the construction of the constrained instanton action was pursued by Dorey and coworkers, and was completed quite recently. The definitive reference is the review article by Dorey, Hollowood, Khoze and Mattis[51]. Hence, in principle, we are able to reproduce the prepotential Seiberg and Witten by some straightforward but tedious integration. However, the formulae for the measure are highly nontrivial and the direct integration is nearly impossible. A further trick was necessary to carry out the integration. The trick was the use of localization. Historically, Flume, Poghossian and Storch [10] and Hollowood[11, 12] were the first to notice that the localization is applicable to the instanton calculation of $\mathcal{N} = 2$ super Yang-Mills equation. They found this by examining carefully the action of the constrained instanton and showed that it can be cast into the framework of cohomological field theory. Later, Nekrasov found[13] that there is an easy way to see that the localization is applicable to the problem at hand. He also obtained the complete formulae for the instanton effect. One of the merit of his method is that we only need an understanding of the geometry of the instanton moduli space and we can bypass the determination of the constrained instanton action. We follow in this chapter the approach taken by Nekrasov and calculate the prepotential using the localization. For a more mathematical exposition, we recommend the reader to consult a good review by H. Nakajima and K. Yoshioka [52].

4.1 Five dimensional supersymmetry

We first review a few basic fact on the five dimensional $\mathcal{N} = 1$ supersymmetry, since Nekrasov's method can be most clearly understood from five-dimensional viewpoint. For the exposition of five dimensional rigid supersymmetry using four-dimensional superfields, we refer the reader to [53] and references therein.

Let us recall the smallest spin representation S of $SO(4,1)$ is of complex dimension four, which can be represented by a four-component Dirac spinor which we are used to with ordinary γ matrices $\gamma^0, \ldots, \gamma^3$ and γ^5. S is isomorphic to its complex conjugate, that is, we have a conjugate-linear norm-preserving map $C : S \to S$. Its square is, however, equal to -1. This property is phrased in physics literature as the representation is pseudo Majorana, or symplectic Majorana. Hence the smallest possible supersymmetry in five dimensions has eight supersymmetries. It gives after a Kaluza-Klein reduction $\mathcal{N} = 2$ supersymmetry in four dimensions. It is convenient for our purpose to decompose the supercharge under the little group $SO(4) \sim SU(2) \times SU(2)$ and write

$$Q_a = \begin{pmatrix} Q^1_\alpha \\ \epsilon_{\dot\alpha\dot\beta}(Q^2_{\dot\beta})^\dagger \end{pmatrix}. \tag{4.2}$$

The most general commutation relation including central charges is, under this decomposition,

$$\{Q^i_a, Q^j_b\} = \epsilon_{ab}\epsilon^{ij}(Z + iP_5), \tag{4.3}$$

$$\{Q^i_a, Q^{i\dagger}_b\} = \gamma^\mu_{ab} P_\mu. \tag{4.4}$$

Here Z is a hermitean central charge. Comparing these commutation relations with that of four dimensional $\mathcal{N} = 2$ supersymmetry (3.18), we find that the imaginary part of the four-dimensional central charge comes from the momentum along the fifth direction.

There are two kinds of multiplets, vector multiplets and hypermultiplets, in the five dimensional $\mathcal{N} = 1$ supersymmetry. They descend to multiplets of the same name under Kaluza-Klein reduction. The spin content of the vector multiplet is slightly different from its counterpart in four dimensions, since five dimensional gauge field has three on-shell degrees of freedom. The scalar contained in the multiplet is real, and is combined with the Wilson line along the fifth direction to become a complex scalar in four dimension.

4.2 Graviphoton background and Ω background

4.2.1 Graviphoton background

Consider an eight-susy field theory on a five dimensional background

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + (dx^5 + A_\mu dx^\mu), \tag{4.5}$$

where $\mu, \nu = 0, 1, 2, 3$, the curvature of A_μ, $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$, is independent of x^μ, and the circumference of the fifth direction is β. That is, circles in the fifth direction is fibered over the four dimensional Euclidean space, and the circles are glued together with constant curvature. Let us call this background geometry the graviphoton background, because the field A_μ is usually called the graviphoton when it is dynamical. We mainly consider the case where $F_{\mu\nu}$ is self-dual. In that case the background preserves half of the supersymmetry.

A supersymmetric theory on this background, in its low energy limit, contains many BPS multiplets with various spin. They can be thought of as an outcome of quantization of solitons in the high energy theory. They can be classified according to the representation of the little group $SO(4) \simeq SU(2)_L \times SU(2)_R$ and their central charges related to various $U(1)$ gauge groups in the theory.

Gopakumar and Vafa showed in [54, 55] that an BPS multiplet with the left spin content

$$I_r = \left((\frac{1}{2}) \oplus 2(0) \right)^{r+1} \tag{4.6}$$

and with central charge a contributes to the prepotential by the amount

$$\mathscr{F}_r(a) = \sum_{k>0} \frac{1}{k} (2 \sinh \frac{kF}{2})^{2r-2} \exp(-ka) \tag{4.7}$$

where F is the magnitude of the field strength. This equation can be proved using the Fock-Schwinger proper time method. Another convenient basis of the left spin content is

$$C_j = \left((\frac{1}{2}) \oplus 2(0) \right) \otimes (\text{a state with } J_L^3 = j) \tag{4.8}$$

In this basis, the contribution to the prepotential becomes

$$\mathscr{F}_j(a) = \sum_{k>0} \frac{1}{k} \frac{1}{(2 \sinh kF/2)^2} \exp(-k(a + 2jF)) \tag{4.9}$$

$$= \sum_{n>0} \log \left(1 - e^{-(a+2jF+nF)} \right). \tag{4.10}$$

Since the prepotential is an quantity protected by supersymmetry and receives contributions only from states annihilated by half of the supersymmetry, i.e. BPS states, the exact prepotential of the low energy theory is given by

$$\mathscr{F} = \sum_{i,r} N_{i,r} \sum_{k>0} \frac{1}{k}(2\sinh\frac{kF}{2})^{2r-2}\exp(-ka_i) \tag{4.11}$$

where $N_{i,r}$ is the number of multiplets with central charge a_i and spin content I_r. $N_{i,r}$ is called the Gopakumar-Vafa invariants of the theory. The prepotential has another expansion

$$\mathscr{F} = F^{-2}\mathscr{F}^{(0)} + \mathscr{F}^{(1)} + F^2\mathscr{F}^{(2)} + F^4\mathscr{F}^{(3)} + \cdots \tag{4.12}$$

and the prepotential in the absence of graviphoton should be identified with $\mathscr{F}^{(0)}$. A remarkable prediction of this argument is that the low energy prepotential, when expanded by the sinh functions as in equation (4.11), will have integer coefficients in the expansion. This greatly constrains the form of prepotential when we have other means of calculating it. For more detailed exposition, we recommend the lecture note by R. Gopakumar [56]. We end this section by mentioning that in order to obtain the genuine prepotential in four dimension one further needs to take the controlled limit $\beta \to 0$.

4.2.2 Ω-background

Nekrasov considered in [13] a five-dimensional geometry

$$ds^2 = (dx^\mu + A_\mu dx^5)^2 + dx^5, \tag{4.13}$$

closely related to the graviphoton geometry (4.5). Here the fifth direction is a circle of circumference β, and the curvature of A_μ is anti-self-dual and independent of x^μ. We denote the magnitude by F. This geometry can be pictorially described as in figure 4.1. That is, four dimensional Euclidean space is fibered over a segment of length β, and the edges are identified using a $SO(4)$ rotation $\exp(i\beta F_{\mu\nu}J^{\mu\nu})$, where $J^{\mu\nu}$ are the generators of $SO(4)$.

We consider a supersymmetric field theory on this background. This time we calculate the partition function of the theory rather than its prepotential. Let us canonically quantize the theory, considering the fifth direction dx^5 as the time direction. Then, in the Hamiltonian formalism, the partition function can be schematically written as

$$Z = \mathrm{Tr}(-)^F e^{-\beta H}\exp(i\beta F_{\mu\nu}J^{\mu\nu}). \tag{4.14}$$

where H is the total Hamiltonian of the field theory. This is none other than the equivariant index of the system. $\exp(i\beta F_{\mu\nu}J^{\mu\nu})$ commutes with half of the supersymmetry when the curvature $F_{\mu\nu}$ is self-dual. Thus, the partition function Z receives contributions only from the states annihilated by those supersymmetry. These states are precisely what contributed to the prepotential of the theory put on the graviphoton background.

These consideration reveals us that the partition function can be written as the infinite product of the form

$$Z = \prod\prod Z_r(a_i)^{N_{i,r}} \tag{4.15}$$

where $N_{i,r}$ is the same Gopakumar-Vafa invariants we discussed above. This tells us how to calculate graviphoton-corrected prepotential of the theory. We first somehow calculate

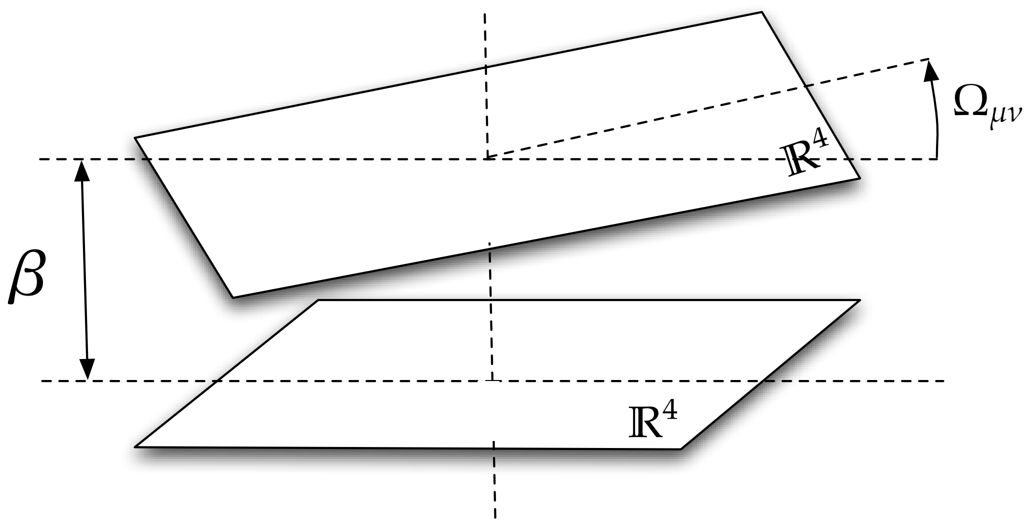


Figure 4.1: Ω-background

the partition function on the Ω-background. Secondly we read off from the expansion of Z the Gopakumar-Vafa invariant. We finally get the desired prepotential by putting these Gopakumar-Vafa invariants into the equation (4.11).

A fortunate coincidence is that the contribution to the partition function from a BPS multiplet with spin content I_r is just the exponential of the contribution to the prepotential from the same BPS multiplet, that is

$$Z_r(a) = \exp(-\mathscr{F}_r(a)) \tag{4.16}$$

This tells us that

$$\mathscr{F}_{\text{graviphoton-corrected}} = \log Z_{\text{on }\Omega\text{-background}}. \tag{4.17}$$

We derive the relation (4.16) in the next subsection.

4.2.3 Contribution of a hypermultiplet

We change the basis from I_r to C_j and show that the contribution to the partition function from C_j is

$$Z_j(a) = \exp(-\mathscr{F}_j(a)) = \prod_{n>0}(1 - e^{-a-2jF}e^{-nF})^{-n}. \tag{4.18}$$

This should be formally equal to the equivariant index

$$\mathrm{Tr} e^{-\beta H} e^{\Omega_{\mu\nu} J_{\mu\nu}}. \tag{4.19}$$

of the system. The target space of the supersymmetric quantum mechanical system is the space of square integrable functions on the spatial slice $\mathbb{R}^4$. It is, however, a difficult quantity to calculate, since the Hamiltonian H has a highly continuous spectrum. A bit of noncommutativity is useful for taming this continuity. Another method is to put the system on $\mathbb{CP}^2$. These two method yield the same result. We take the first path and calculate $Z_{j=0}(0)$.

Combine the four coordinates into two complex variables $z_{1,2}$ and introduce non-commutativity $[z_1, \bar{z}_1] = [z_2, \bar{z}_2] = i\epsilon$. The calculation of the spectrum reduces to the well-known Landau level problem, and we find the decomposition

$$\mathcal{L}^2(\mathbb{C}^2) = \mathbf{1} \oplus (\mathbf{2} \oplus \mathbf{2}) \oplus \cdots \oplus \underbrace{(\mathbf{n} \oplus \cdots \oplus \mathbf{n})}_{n \text{ times}} \oplus \cdots \tag{4.20}$$

where $\mathbf{n}$ denotes complex one-dimensional space with J_L^3 eigenvalue n. Hence, using the fixed point formula,

$$Z_{j=0}(0) = \prod_{n>0} \left(\frac{1}{e^{nF/2} - e^{-nF/2}} \right)^n = \prod_{n>0} \left(\frac{1}{1 - e^{-nF}} \right)^n \tag{4.21}$$

where we used in the second equality a zeta function regularization

$$1^2 + 2^2 + 3^2 + \cdots = \zeta(-2) = 0. \tag{4.22}$$

Extension to the higher spin representation and inclusion of the central charge is straightforward.

4.3 Hilbert scheme of points on $\mathbb{C}^2$

In the last section we carried out the calculation of the partition function in a second quantized setup. The same result can be obtained in a first quantized framework. In a first-quantized framework, the system is thought of as a collection of particles and anti-particles. The calculation of the partition function is 'localized' by the supersymmetry to the configuration space of BPS states. A BPS configuration is a collection of particles only, since an anti-particle respects the other half of the supersymmetry and particle-antiparticle pair breaks all of the supersymmetry. As the particles are indistinguishable from each other, the configuration space of k particles is

$$S^k\mathbb{C}^2 \equiv (\mathbb{C}^2)^k / \mathfrak{S}_k. \tag{4.23}$$

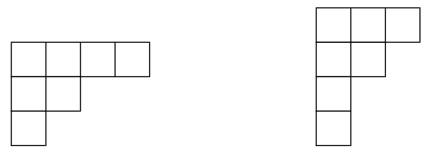
Hence, the partition function should be

$$Z = \sum e^{-ka} \mathrm{Ind}_g S^k \mathbb{C}^2. \tag{4.24}$$

However, the space $S^k\mathbb{C}^2$ is highly singular and reliable calculation of the equivariant index is difficult. It is known that there is a good resolution of the space $S^k\mathbb{C}^2$ denoted by $(\mathbb{C}^2)^{[n]}$, called the Hilbert scheme of n-points on $\mathbb{C}$. Let us now collect some relevant fact about the Hilbert scheme of points.

4.3.1 Definition of $(\mathbb{C}^2)^{[n]}$

We should comment that the explanation given here is not very rigorous and we recommend the reader to consult [57] for mathematically precise expositions. Let us parametrize $\mathbb{C}^2$ by two variables x, y and consider n distinct points $p_1 = (x_1, y_1), \ldots, p_2 = (y_n, x_n)$ on $(\mathbb{C}^2)$. We associate to the set $\{p_1, \ldots, p_N\}$ polynomial functions $P(x, y)$ of x and y vanishing at all p_i. The set of such functions form an ideal $I_{\{p_1,\ldots,p_n\}}$ of the ring $\mathbb{C}[x, y]$ Note that the quotient



$Y = (4 \geq 2 \geq 1)$ $\qquad$ $Y^D = (3 \geq 2 \geq 1 \geq 1)$

Figure 4.2: Young tableaux

$\mathbb{C}[x,y]/I$ is n-dimensional. Hence we have obtained a map from non-singular points in $S^n\mathbb{C}^2$ to codimension n ideals of $\mathbb{C}[x,y]$. Let us consider all the codimension n ideals in $\mathbb{C}[x,y]$, and denote it by $(\mathbb{C}^2)^{[n]}$. It is called the Hilbert scheme of n points on $\mathbb{C}^2$.

Now that we have understood the definition of $(\mathbb{C}^2)^{[n]}$, we go on to the calculation of the equivariant index of them. As the Cartan torus of $SO(4)$ acts by $U(1)$ phase rotation of x and y, a fixed point I in $(\mathbb{C}^2)^{[n]}$ corresponds to an codimension n ideal of $\mathbb{C}[x,y]$ which is invariant under the action $x \to e^{i\theta}x$ and $y \to e^{i\theta'}y$. For such an ideal, the representative of $\mathbb{C}[x,y]/I$ consists of monomials $x^m y^n$. Let us denote the set of such monomials by Y. Y should satisfy a condition since I is an ideal, i.e. $xI \subset I$ and $yI \subset I$. The condition on the ideal reflects to the fact that if $x^m y^n$ is included in Y, then $x^{m'}y^{n'}$ with $m' < m$ and $n' < n$ also is included in Y. Such a set of pair of integers (m,n) form a Young tableau. We introduce here for later use some notations for the Young tableaux. We identify a Young tableau with a partition and write $Y = (y_1 \geq y_2 \geq y_3 \cdots)$. We write Y^D the dual tableau of Y obtained by reflecting along the diagonal(see figure 4.2). We denote the box in the j-th column of the i-th row as the box (i,j). We write $(i,j) \in Y$ if $0 \leq j \leq y_i$. From a Young tableau T one can construct a codimension n ideal I by

$$I = \text{ideal generated by all } x^m y^n \text{ where } (m,n) \notin T \tag{4.25}$$

Thus we established the correspondence between a Young tableau and a fixed points in $(\mathbb{C}^2)^{[n]}$.

4.3.2 Another realization of $(\mathbb{C}^2)^{[n]}$

In order to calculate the contribution to the equivariant index from each of the fixed points, we need understand the tangent space of $(\mathbb{C}^2)^{[n]}$ at the fixed point. The definition using the ideals we reviewed above is not suitable for this purpose. Hence we develop another realization of $(\mathbb{C}^2)^{[n]}$ here.

First let us temporarily introduce a notation

$$M_n = \frac{\left\{ \begin{array}{c} B_1, B_2 \in \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n), \\ I \in \mathrm{Hom}(\mathbb{C},\mathbb{C}^n) \end{array} \middle| \begin{array}{ll} \text{i)} & [B_1,B_2] = 0 \\ & \text{there is no proper subspace in} \\ \text{ii)} & \mathbb{C}^n \text{ containing } \mathrm{Im}I \text{ and closed} \\ & \text{under the action of } B_1 \text{ and } B_2 \end{array} \right\}}{GL(n,\mathbb{C})} \tag{4.26}$$

where $GL(n,\mathbb{C})$ acts on B_1 and B_2 by conjugation and on I by right multiplication. The condition ii) is called the stability condition. We now prove that there is a one-to-one correspondence between M_n and $\mathbb{C}^{[n]}$.

From $\mathbb{C}^{[n]}$ to M_n: for a codimension n ideal $I \subset \mathbb{C}[x,y]$, consider a vector space $V = \mathbb{C}[x,y]/I$. The dimension is by definition n, hence $V \simeq \mathbb{C}^n$. Multiplication by x and y is well-defined on V since $xI \subset I$ and $yI \subset I$ from the defining property of an ideal. Thus, x and y define two elements in $\mathrm{Hom}(V,V)$, and the two trivially commutes. Finally, we have a map sending $1 \in \mathbb{C}$ to $[1] \in V$. In order to see that these data comprise just an element of M_n, we need to check the stability. But this is also trivial as any element of V can be generated by repeated action of x and y.

From M_n to $\mathbb{C}^{[n]}$: From the stability condition, the image of I is one dimensional. We consider a map ϕ from $\mathbb{C}[x,y]$ to $\mathbb{C}^n$ defined by

$$\mathbb{C}[x,y] \ni P(x,y) \mapsto P(B_1,B_2)I(1) \in \mathbb{C}^n \tag{4.27}$$

$P(B_1,B_2)$ is well-defined thanks to the condition i). The image of ϕ is closed under the action of B_1 and B_2 and includes $I(1)$, $\mathrm{Im}\phi = \mathbb{C}^n$ from stability. Hence the kernel of ϕ is a codimension n ideal in $\mathbb{C}[x,y]$.

4.3.3 Yet another realization of $(\mathbb{C}^2)^{[n]}$

Let us construct a modified version of the linear algebraic construction of $(\mathbb{C}^2)^{[n]}$ presented above. The construction reveals the striking equivalence of the Hilbert scheme of points and non-commutative $U(1)$ instantons.

Let us denote by N_n the following space:

$$N_n = \frac{\left\{ \begin{array}{c} B_1,B_2 \in \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n), \\ I \in \mathrm{Hom}(\mathbb{C},\mathbb{C}^n), \\ J \in \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}) \end{array} \middle| \begin{array}{ll} \text{i)} & [B_1,B_2]+IJ=0 \\ & \text{there is no proper subspace in} \\ \text{ii)} & \mathbb{C}^n \text{ containing } \mathrm{Im}I \text{ and closed} \\ & \text{under the action of } B_1 \text{ and } B_2 \end{array} \right\}}{GL(n,\mathbb{C})} \tag{4.28}$$

This definition differs from that of equation (4.26) just in the appearance of J. We prove that nonetheless there is a natural one-to-one map between M_n and N_n.To show this statement, it suffices to prove that J is a zero map. Notice that from the stability condition, all element of $\mathbb{C}^n$ can be obtained by repeated application of B_1 and B_2 to $I(1)$. Hence we need only to show that applying J to an element of the form

$$B_{i_1}B_{i_2}\cdots B_{i_k}I(1) \tag{4.29}$$

results in zero. We show this in induction on k.

For $k=0$: as JI is a map from $\mathbb{C}$ to $\mathbb{C}$, hence it coincides with its trace. Hence

$$JI = \mathrm{tr}(JI) = \mathrm{tr}(IJ) = \mathrm{tr}([B_1,B_2]) = 0. \tag{4.30}$$

Suppose that we have shown the claim for $k < m$. Consider an expression of the form $JB_{i_1}\cdots B_2B_1\cdots B_{i_m}$. Using the condition $IJ = -[B_1,B_2]$, we can rewrite this as

$$JB_{i_1}\cdots B_2B_1\cdots B_{i_m} = JB_{i_1}\cdots IJ\cdots B_{i_m} + JB_{i_1}\cdots B_1B_2\cdots B_{i_m}. \tag{4.31}$$

The first term in the right hand side is zero from the induction hypothesis. Hence we need to only show that

$$JB_1^aB_2^bI = 0. \tag{4.32}$$

This can be proved by an extension of the argument for $k = 0$. We can rewrite, by using various identities,

$$JB_1^a B_2^b I = \mathrm{tr}(JB_1^a B_2^b I) = \mathrm{tr}(B_1^a B_2^b IJ) \tag{4.33}$$

$$= -\mathrm{tr}(B_1^a B_2^b [B_1, B_2]) = -\mathrm{tr}([B_1^a B_2^b, B_1] B_2) \tag{4.34}$$

$$= -\sum_{c+d=b} \mathrm{tr}(B_1^a B_2^c [B_2, B_1] B_2^d) = -\sum_{c+d=b} \mathrm{tr}(B_2^d B_1^a B_2^c [B_2, B_1]) \tag{4.35}$$

$$= -\sum_{c+d=b} \mathrm{tr}(B_2^d B_1^a B_2^c IJ) = -\sum_{c+d=b} JB_2^d B_1^a B_2^c I. \tag{4.36}$$

Since $JB_2^d B_1^a B_2^c = JB_1^a B_2^b$ by applying the identity shown in (4.31), we get $JB_1^a B_2^b I = -bJB_1^a B_2^b I$. Thus $JB_1^a B_2^b I$=0. This completes the proof.

A noticeable fact is that the linear data before taking the quotient with respect to $GL(n, \mathbb{C})$ is just the linear ADHM data for '$U(1)$-instanton'. Moreover, the condition $[B_1, B_2] + IJ = 0$ is just the condition $\mu_{\mathbb{C}} = 0$. In the next paragraph we digress a little about the relationship between the two.

4.3.4 On symplectic quotient and holomorphic quotient

Consider a complex manifold M equipped with an action of a compact Lie group G. We can take two different quotients from M and G. The first one is

$$M/G_{\mathbb{C}}, \tag{4.37}$$

where we divide the space by the action of the complexified form of G. The second is defined with the help of the Hamiltonian μ generating the flow of G:

$$\mu^{-1}(a)/G \tag{4.38}$$

where M is endowed with a symplectic form naturally defined by the complex structure. It is known that there is a close relationship between the two. Firstly we review the case for $a = 0$ and then we state the correspondence for nonzero a.

The relation for the case $a = 0$ is implicitly known since the earliest work on supersymmetry. μ is the D-term in the physics language. Hence $\mu^{-1}(0)/G$ is the space of vacua for supersymmetric gauge theory. On the other hand, we know that in the superspace formalism the gauge group is enhanced to $G_{\mathbb{C}}$. Thus the space of vacua can also be written as $M/G_{\mathbb{C}}$. For example see chapter VIII of the textbook by Wess and Bagger[58]. For almost all point, we can make one-to-one map between $M/G_{\mathbb{C}}$ and $\mu^{-1}(0)/G$. We may disregard the difference for the most purpose. However, the fixed points often lie in the subtle, somewhat singular points, we need to proceed carefully. The precise statement for the case where M is a vector space is

Theorem There is a bijection between the set

$$\mu^{-1}(0)/G \qquad \text{and} \qquad M//G_{\mathbb{C}} \tag{4.39}$$

where the "geometric invariant theory" quotient $M//G_{\mathbb{C}}$ is defined through the equivalence relation

$$x \sim y \qquad \text{if and only if} \qquad \overline{G_{\mathbb{C}} x} \cap \overline{G_{\mathbb{C}} y} \neq \emptyset \tag{4.40}$$

for $x, y \in M$. Moreover, for any point x we can find x_0 such that $G_{\mathbb{C}} x_0 = \overline{G_{\mathbb{C}} x}$.

This can be proved by chasing the behavior of the moment map μ under the complexified gauge group $G_{\mathbb{C}}$. The reader will find a readable account of the theorem in the work of Luty and Taylor[59].

In order to state the corresponding result for nonzero a, that is for nonzero Fayet–Iliopoulos term in the physics language, we need to prepare a little. The moment map $\mu(x)$ determines the Hamiltonian for the G action through the combination

$$H(g) = \langle \mu(x), g \rangle \tag{4.41}$$

where g is some generator of the Lie algebra $\mathfrak{g}$ of G. Hence μ is a map from M to $\mathfrak{g}^*$. Thus, Fayet–Iliopoulos term a can be seen as a map $a : u(1)^* \to \mathfrak{g}^*$. In other words, this determines a one dimensional representation of G through

$$z \to e^{i\langle a, g \rangle} z. \tag{4.42}$$

We endow the space $V \times \mathbb{C}^\times$ with G action by

$$g(x, z) \to (gx, e^{i\langle a, g \rangle} z) \tag{4.43}$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. After these preparation, we can state the correspondence.

Theorem There is a bijection between the set

$$\mu^{-1}(a)/G \qquad \text{and} \qquad (V \times \mathbb{C}^\times)//_a G_{\mathbb{C}} \tag{4.44}$$

where the quotient $M//_a G_{\mathbb{C}}$ is defined through the equivalence relation

$$x \sim y \qquad \text{if and only if} \qquad \overline{G_{\mathbb{C}}(x, z)} \cap \overline{G_{\mathbb{C}}(y, z)} \neq \emptyset \tag{4.45}$$

for $x, y \in M$. Moreover, for any point x we can find x_0 such that $G_{\mathbb{C}}(x_0, z) = \overline{G_{\mathbb{C}}(x, z)}$. The physical interpretation of this theorem will be discussed somewhere by the author.

We can show that the construction of the space N_n, equation (4.28), is none other than the quotient $\mu_{\mathbb{C}}(0)//_a GL(n)$. The theorem above equates this to $\mu_{\mathbb{C}}(0) \cap \mu_{\mathbb{R}}(a)/U(n)$. We see that this is precisely the hyperkähler quotient description of non-commutative $U(1)$ instantons.

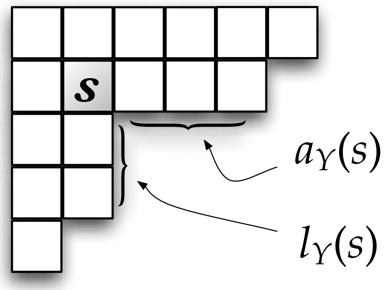
4.3.5 Residues at fixed points

Using these linear algebraic description, we can easily work out the residue at the fixed points. The residue is determined by the action of

$$(e^{i\theta_1}, e^{i\theta_2}) \in U(1)^2 \subset SO(4) \tag{4.46}$$

on the tangent space at the fixed point.

Let us first study the $U(1)^2$ action on the linear data (B_1, B_2, I, J). We denote by t_1 and t_2 the one dimensional representation of $U(1)^2$ respectively acted by $e^{i\theta_1}$ and $e^{i\theta_2}$. From the correspondence to the codimension n ideal, B_1 and B_2 transform respectively in t_1 and t_2. From the constraint $[B_1, B_2] + IJ = 0$, IJ should transform in $t_1 t_2$. The transformation of I and J themselves does not make any difference to the result. We take I to transform trivially and J to transform in the representation $t_1 t_2$.



$Y = (6 \geq 5 \geq 2 \geq 2 \geq 1)$

Figure 4.3: a Young tableau and its hook lengths

Consider a fixed point Y on $(\mathbb{C}^2)^{[n]}$ and its corresponding linear data (B_1, B_2, I, J). The fact that Y is fixed under $t \in U(1)^2$ means that there is a $GL(n)$ transformation $\phi(t)$ such that

$$\phi(t)B_1\phi(t)^{-1} = t_1 B_1, \tag{4.47}$$
$$\phi(t)B_2\phi(t)^{-1} = t_2 B_2, \tag{4.48}$$
$$\phi(t)I = I, \tag{4.49}$$
$$J\phi(t) = t_1 t_2 J. \tag{4.50}$$

This map ϕ defines a homomorphism $U(1)^2 \to GL(n)$ and makes $\mathbb{C}^n$ into a representation of $U(1)^2$. The representation content is determined from the correspondence to the codimension n ideal, and the result is

$$\mathbb{C}^n = \bigoplus_{(i,j)\in Y} t_1^{1-j} t_2^{1-i}. \tag{4.51}$$

We now move on to the study of the tangent space. We need to introduce some notation about the Young tableaux to state the results. Define for a Young tableau its arm length and leg length by

$$a_Y(s) = y_i - j, \tag{4.52}$$
$$l_Y(s) = y_j^D - i. \tag{4.53}$$

where $s = (i, j)$. The definition is depicted in figure 4.3. The claim is that for a fixed point labeled by a Young tableau Y,

$$T(\mathbb{C}^2)^{[n]}\Big|_Y = \bigoplus_{s\in Y} \left(t_1^{l(s)+1} t_2^{-a(s)} + t_1^{-l(s)} t_s^{a(s)+1} \right). \tag{4.54}$$

We provide the salient part of the proof. For a data $(B_1, B_2, I, 0)$ corresponding to a Young tableaux Y, consider the complex

$$\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \xrightarrow{p} \begin{matrix} \mathrm{Hom}(\mathbb{C}^n, (t_1 \oplus t_2) \otimes \mathbb{C}^n) \\ \oplus \\ \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n) \\ \oplus \\ \mathrm{Hom}(\mathbb{C}^n, t_1 t_2 \mathbb{C}) \end{matrix} \xrightarrow{q} t_1 t_2 \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n), \tag{4.55}$$

with

$$p(\xi) = \begin{pmatrix} [\xi, B_1] \\ [\xi, B_2] \\ \xi I \\ 0 \end{pmatrix}, \qquad q\begin{pmatrix} b_1 \\ b_2 \\ i \\ j \end{pmatrix} = [B_1, b_2] + [b_1, B_2] + Ij. \tag{4.56}$$

Since p is the differential of the $GL(n)$ action and q is the differential of the constraint $[B_1, B_2] + IJ = 0$, the tangent space at Y can be identified with $\mathrm{Ker}q/\mathrm{Im}p$.

Hence, the $U(1)^2$ module structure can be easily written down as a virtual representation

$$T(\mathbb{C}^2)^{[n]}\Big|_Y = (t_1 \oplus t_2 \ominus t_1 t_2 \ominus 1)\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n) \oplus t_1 t_2 \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}). \tag{4.57}$$

To simplify the expression (4.57) to the desired form (4.54) is a straightforward but tedious combinatorial exercise. We refer the reader to the lecture notes by H. Nakajima[57].

Nekrasov obtained another suggestive form of the representation content of the tangent space for diagonal $U(1)$ subgroup $U(1) \ni e^{i\theta} \mapsto (e^{i\theta}, e^{-i\theta}) \in U(1)^2$. It is

$$T(\mathbb{C}^2)^{[n]}\Big|_Y = \bigoplus_{i \neq j} (t^{y_i - y_j + j - i} \ominus t^{j-i}). \tag{4.58}$$

for a Young tableau $Y = (y_1 \geq y_2 \geq \cdots)$. The equivalence of the expression (4.54) and (4.58) can be proved with some efforts.

4.3.6 Contribution of a hypermultiplet

From the data, one can finally obtain the partition function of a hypermultiplet in the Ω background. Applying the fixed point theorem using the data obtained in (4.58), the answer is

$$Z_{j=0}(0) = \sum_Y \prod_{s \in Y} \frac{1}{1 - t_1^{l(s)+1} t_2^{-a(s)}} \frac{1}{1 - t_1^{-l(s)} t_2^{a(s)-1}}. \tag{4.59}$$

The equality of the infinite series (4.59) and the infinite product (4.21) can be proved by utilizing the free fermions[60, 61]. Let us briefly review the derivation. We restrict the argument for the case when $t_1 = t_2^{-1} = q$.

Firstly, introduce the vertex operator on a boson Fock space

$$\Gamma_\pm(t_n) = \exp(\sum_n t_n \alpha_{\pm n}). \tag{4.60}$$

Γ_+ is the conjugate of Γ_- and they satisfy the relation

$$\Gamma_+(t_n)\Gamma_-(s_n) = \exp(\sum_n n t_n s_n)\Gamma_-(s_n)\Gamma_+(t_n). \tag{4.61}$$

We further introduce the notation

$$V_\pm(x_i) = \Gamma_\pm(t_n = \frac{1}{n}\sum x_i^n). \tag{4.62}$$

They satisfy the commutation relation

$$V_+(x_i)V_-(y_i) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_i} V_-(y_i)V_+(x_i). \tag{4.63}$$

Secondly, recall that a Young tableau Y naturally define a free fermion state $|Y\rangle$. We define the quantum dimension of a Young tableau Y by the formula

$$\dim_q Y = \langle Y| V_-(x_i = q^{-i+1/2}) |0\rangle . \tag{4.64}$$

We can show that the equation (4.59) can be rewritten as

$$Z_{j=0}(0) = \sum_Y |\dim_q Y|^2 = \langle 0| V_+(x_i) |Y\rangle \langle Y| V_-(x_i) |0\rangle . \tag{4.65}$$

Since the bases $|Y\rangle$ span a complete orthonormal set, we can evaluate the above equation to

$$= \langle 0| V_+(x_i) V_-(x_i) |0\rangle \tag{4.66}$$

$$= \prod_{i,j\geq 1} \frac{1}{1 - x_i x_j} \langle 0| V_-(x_i) V_+(x_i) |0\rangle \tag{4.67}$$

$$= \prod_{n\geq 1} \left(\frac{1}{1 - q^{-n}} \right)^n . \tag{4.68}$$

This matches with the equation (4.21).

4.4 Super Yang-Mills on the Ω background

Let us utilize the considerations above to the calculation of the exact prepotential of Yang-Mills theories with eight supersymmetries. The argument in the previous section tells us that, in order to calculate the graviphoton-corrected prepotential for the super Yang-Mills theory, it just suffices to put the theory on the Ω background and evaluate its equivariant index. Since the super Yang-Mills theory has some moduli and the prepotential is a function of them, we need to encode the information to the five-dimensional setup.

Four dimensional $\mathcal{N} = 2$ pure $SU(N)$ Yang-Mills theory has $N - 1$ complex parameters as its moduli. They correspond to $N - 1$ real scalars and $N - 1$ Wilson lines around the fifth direction when the theory is considered as coming from the Kaluza-Klein reduction of the five-dimensional theory. These are combined into $N - 1$ complex variables and the prepotential is a holomorphic function of them because of the supersymmetry. Thus, it suffices to introduce only the vacuum expectation values for real scalars or only the Wilson lines. It turns out to be easier to include the Wilson line to our setup. With Wilson lines turned on, the partition function we should compute becomes

$$Z = \mathrm{Tr}(-)^F e^{-\beta H} e^{i\beta \Omega_{\mu\nu} J^{\mu\nu}} e^{i\beta a_i J_i}, \tag{4.69}$$

where J_i is generators of the Cartan subgroup of the global gauge rotations. We can anticipate that a_i turn out to be the special coordinates, since a hypermultiplet with unit charge in five dimensions gives a massive hypermultiplet in four dimensions with mass proportional to the Wilson line. Denote by $g = e^{i\beta\Omega_{\mu\nu}J^{\mu\nu}} e^{i\beta a_i J_i}$ and $T^{N+2} \subset SO(4) \times U(N)$ the Cartan torus.

To extract four-dimensional result, we need to take $\beta \to 0$ with $\Omega_{\mu\nu}$ and a_i fixed. Taking $\Omega_{\mu\nu} \to 0$, we should be able to calculate the exact low-energy prepotential of the $\mathcal{N} = 2$ pure $SU(N)$ Yang-Mills theory. We will see later in this chapter that the outcome exactly matches with the weak-coupling expansion obtained in (3.3.2) . By a further trick, we can see the Seiberg-Witten curve directly emerging in the instanton calculation.

We argued above that in order to obtain the effective prepotential, we have to calculate the $SO(4) \otimes U(n)$ equivariant index of the quantum field theory. We now go on to study the supersymmetric quantum mechanics that calculates the index. Firstly, the configuration space $\supset \mathscr{A}/\mathscr{G}$ can be divided according to the instanton number. Lowest energy states in each topological sector is the (anti-)self-dual configuration. Hence, as a first approximation we take the instanton moduli space as the target space of the quantum mechanical system. Index theorem tells us that on the k-instanton moduli has real $4Nk$ dimension and has real $4Nk$ adjoint fermion zero modes. Thus, the supersymmetric quantum mechanics we consider has $M_{N,k}$ as the target space and the Hilbert space is the sections of the spin bundle of $M_{N,k}$. We can argue that there are no higher order correction to the equivariant index, along the lines presented in [9, 62].

Hence, the instanton contribution to the partition function is that

$$Z_\Omega = \sum_k e^{-\beta\tau k} \mathrm{Ind}_g M_{N,k}. \tag{4.70}$$

The equivariant index of a manifold can be calculated using the fixed point theorems reviewed in section 2.4. Thus, the calculation of the instanton correction has been reduced to the study of the fixed points and the action of g around them.

4.4.1 Enumeration of the fixed points

Let us now study the fixed points and their residues. The manifolds $M_{N,k}$ are obtained by an hyperkähler quotient construction starting from the vector space $\mathbb{X}$ as described in section 1.2.2. First we have to learn how g acts on $\mathbb{X}$.

$SO(4) \times U(N)$ action on $\mathbb{X}$

We concentrate on the action of the Cartan torus $U(1)^2 \times U(1)^N$. We take $U(1)^2 \ni (e^{i\theta_1}, e^{i\theta_2})$ to act on the coordinates $(z_1, z_2) \in \mathbb{R}^4$ as

$$z_1 \to e^{i\theta_1} z_1, \qquad z_2 \to e^{i\theta_2} z_2. \tag{4.71}$$

From the construction of the ASD connection, eq (1.24), we see that B_1 and B_2 should be given the transformation

$$B_1 \to e^{i\theta_1} B_1, \qquad B_2 \to e^{i\theta_2} B_2. \tag{4.72}$$

Hence, the combination IJ should transform as $IJ \to e^{i\theta_1 + i\theta_2} IJ$. The transformation of individual parts I and J is immaterial to the final result and hence we could choose arbitrarily. From the viewpoint of the hyperkähler structure, it seems more natural to assign I and J the transformation $I \to e^{i(\theta_1+\theta_2)/2} I$ and $J \to e^{i(\theta_1+\theta_2)/2} J$. It is more convenient, however, to assign $J \to e^{i(\theta_1+\theta_2)} J$ and leaves $I \to I$.

The $U(N)$ action is straightforward. $W \sim \mathbb{C}^n$ transforms as a fundamental representation under $U(N)$ and $V \sim \mathbb{C}^k$ is left intact. This determines all the transformation properties. We denote the one dimensional representations in the decomposition of W

$$W = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_N \tag{4.73}$$

where $\alpha = e^{i\beta a_i}$.

Fixed points

We study fixed points in $M_{N,k}$ following [15]. A g fixed point in $M_{N,k}$ comes from a $U(k)$ orbit in $\mathbb{X}$ which is fixed by g. Put differently, an element $X \in \mathbb{X}$ descends to a fixed point in $M_{N,k}$ only if there is some element $\phi(g) \in u(k)$ such that

$$gX = \phi(g)X. \tag{4.74}$$

Further, in order to descend to a fixed point in $M_{N,k}$, these should satisfy

$$[B_1, B_2] + IJ = 0. \tag{4.75}$$

It is clear that $\phi(g)$ defines a homomorphism from $U(1)^2 \times U(1)^N$ to $U(k)$.

Recall that the space V naturally carries an action of $U(k)$. We can introduce an action of $U(1)^2 \times U(1)^N$ through the homomorphism ϕ. Let us decompose V into its irreducible representation, i.e. eigenspaces of g:

$$V = V_0 \oplus \bigoplus_i V_i, \tag{4.76}$$

where where $U(1)^N$ acts on V_i as α_i. V_0 is the part of V where $U(1)^N$ acts in some other representation.

The condition (4.74) means that the operators

$$B_1, B_2 : V \to V, \qquad I : W \to V, \qquad J : V \to W. \tag{4.77}$$

do not change $U(1)^N$ representation. An inspection shows that $V_i \oplus W_i$ for $i = 1, \dots, N$ and V_0 themselves form a linear ADHM datum for a fixed point in $M_{1,\dim V_i}$. That is, B_1, B_2, I, J restricted on the respective vector space satisfy the conditions (4.28). From the stability condition, V_0 must be empty. Thus, the analysis is reduced to the classification of the fixed points for $U(1)$ gauge group. We know that instantons in $U(1)$ gauge theory are all singular. Hence the fixed points correspond to the singular gauge configurations. We need to introduce noncommutativity to resolve the singularity.

But notice here that after the resolution of singularities, the fixed point in $M_{1,\dim V_i} \simeq (\mathbb{C}^{[\dim V_i]})$ is none other than the fixed points in $(\mathbb{C}^2)^{[n]}$, the Hilbert scheme of points on $\mathbb{C}^2$. These were already classified in section 4.3.5 ! Hence we immediately obtain the classification of the fixed points. A fixed point can be labeled by a N-tuple of Young tableaux $(Y_1, \dots, Y_N)$.

4.4.2 Residues at Fixed Points; Nekrasov's Formula

Next, we study the action of g on the tangent space at the fixed point. The tangent space can be studied just as in section 4.3.5. Let us first express the tangent space as a cohomology of a chain complex:

$$\mathrm{Hom}(V,V) \xrightarrow{p} \begin{pmatrix} (t_1 \oplus t_2) \otimes \mathrm{Hom}(V,V) \\ \oplus \\ \mathrm{Hom}(W,V) \oplus t_1 t_2 \otimes \mathrm{Hom}(V,W) \end{pmatrix} \xrightarrow{q} t_1 t_2 \mathrm{Hom}(V,V) \tag{4.78}$$

where

$$p(\xi) = \begin{pmatrix} [\xi, B_1] \\ [\xi, B_2] \\ \xi I \\ -J\xi \end{pmatrix}, \qquad q \begin{pmatrix} b_1 \\ b_2 \\ i \\ j \end{pmatrix} = [B_1, b_2] + [b_1, B_2] + Ij + iJ. \tag{4.79}$$

Note that p is the differential of the $U(k)$ action and that q is the differential of the constraint $\mu_{\mathbb{C}} - \zeta_{\mathbb{C}} = 0$ in order to see that $\mathrm{Ker}q/\mathrm{Im}p$ is indeed the tangent space at the fixed points. As the actions of $U(1)^2 \times U(1)^N$ on various factors are trivially known, the module structure of the tangent space is known by adding and subtracting those factors. Firstly, $\mathrm{Ker}q/\mathrm{Im}p$ is decomposed according to the $U(1)^N$ action as $\bigoplus_{i,j}(\mathrm{Ker}q_{ij}/\mathrm{Im}p_{ij})$, where p_{ij} and q_{ij} are the the restriction of p and q onto the subspace:

$$\mathrm{Hom}(V_i, V_j) \xrightarrow{p_{ij}} \begin{array}{c} (t_1 \oplus t_2) \otimes \mathrm{Hom}(V_i, V_j) \\ \oplus \\ \mathrm{Hom}(W_i, V_j) \oplus t_1 t_2 \mathrm{Hom}(V_i, W_j) \end{array} \xrightarrow{q_{ij}} t_1 t_2 \mathrm{Hom}(V_i, V_j). \tag{4.80}$$

The analysis of $\mathrm{Ker}q_{ij}/\mathrm{Im}p_{ij}$ is a direct extension of the analysis presented in section 4.3.5. The result is

$$\mathrm{Ker}q_{ij}/\mathrm{Im}p_{ij} = \alpha_j \alpha_i^{-1} \otimes \left(\bigoplus_{s\in Y_i} (t_1^{-l_{Y_j}(s)} t_2^{a_{Y_i}(s)+1}) \oplus \bigoplus_{s\in Y_j} (t_1^{l_{Y_i}(s)+1} t_2^{-a_{Y_j}(s)}) \right) \tag{4.81}$$

Hence, we finally get the complete five-dimensional partition function:

$$Z = \sum_k q^k \sum_{(Y_1,\ldots,Y_N), \sum \#Y_i = k} \sum_{i,j} \sum_{s\in Y_i \cup Y_j} \frac{1}{1 - \alpha_i \alpha_j^{-1} t_1^{-l_{Y_j}(s)} t_2^{a_{Y_i}(s)+1}} \frac{1}{1 - \alpha_i \alpha_j^{-1} t_1^{l_{Y_i}(s)+1} t_2^{-a_{Y_j}(s)}} \tag{4.82}$$

This can be further simplified to the form originally calculated by Nekrasov, for the special case $t_1 t_2 = 1$:

$$Z = \sum_k q^k \sum_{(Y_1,\ldots,Y_N), \sum \#Y_i = k} \sum_{i,j} \sum_{s\in Y_i \cup Y_j}$$
$$\frac{1}{\sinh \frac{\beta}{2}(a_i - a_j + \epsilon(-l_{Y_j}(s) - a_{Y_i}(s) - 1))} \frac{1}{\sinh \frac{\beta}{2}(a_i - a_j + \epsilon(l_{Y_i}(s) + a_{Y_j}(s) + 1))} \tag{4.83}$$
$$= \sum_k q^k \sum_{(Y_1,\ldots,Y_N), \sum \#Y_i = k} \sum_{(i,m)\neq(j,n)} \frac{\sinh \frac{\beta}{2}(a_i - a_j + \epsilon(y_{i,n} - y_{j,m} + m - n))}{\sinh \frac{\beta}{2}(a_i - a_j + \epsilon(m-n))} \tag{4.84}$$

Taking the four-dimensional limit $\beta \to 0$ with a_i and ϵ fixed, we get the result

$$Z = \sum_k \Lambda^{2Nk} \sum_{(Y_1,\ldots,Y_N), \sum \#Y_i = k} \sum_{(i,m)\neq(j,n)} \frac{(a_i - a_j + \epsilon(y_{i,n} - y_{j,m} + m - n))}{(a_i - a_j + \epsilon(m-n))} \tag{4.85}$$

4.4.3 Comparison against Seiberg-Witten theory

We have finally obtained the all-instanton result for the prepotential. Let us compare it against the result obtained from the Seiberg-Witten curves, section 3.3.4. Firstly we have to extract the prepotential from equation (4.85). As is argued in section 4.2.1, the answer is

$$\mathscr{F} = \lim_{\epsilon\to 0} \epsilon^2 \log Z. \tag{4.86}$$

We can calculate the instanton correction explicitly. Up to two instantons, the answers are

$$\mathscr{F}^{(1)} = \frac{1}{2}\sum_{l}\prod_{k\neq l}\frac{1}{(a_k-a_l)^2}, \tag{4.87}$$

$$\begin{aligned}\mathscr{F}^{(2)} = &\frac{1}{4}\sum_{l}\sum_{k\neq l,\, m\neq l}\frac{1}{a_k-a_l}\frac{1}{a_m-a_l}\prod_{k\neq l}\frac{1}{(a_k-a_l)^2}\\ &+\frac{3}{8}\sum_{l}\sum_{k\neq l}\frac{1}{(a_k-a_l)^2}\prod_{k\neq l}\frac{1}{(a_k-a_l)^2}\\ &+\frac{1}{4}\sum_{l\neq m}\frac{1}{(a_l-a_m)^2}\prod_{k\neq l}\frac{1}{(a_k-a_l)^2}\prod_{k\neq m}\frac{1}{(a_k-a_m)^2}.\end{aligned} \tag{4.88}$$

These expressions exactly agree with the result obtained in section 3.3.4, equation (3.91).We will see in the following sections that they agree to each other to arbitrarily high order by extracting the curve from the Nekrasov's formula.

4.5 Integral representation on the ADHM data

There is another method of computation of the graviphoton-corrected prepotential. We review in this section the method pioneered by Moore, Nekrasov and Shatashvili[63]. For a mathematically precise exposition, we refer the reader to [64].

First, take the four-dimensional limit and express the partition function as

$$Z^{\text{4 dim.}}_{\text{Nekrasov}} = \sum_k \Lambda^{4Nk}\sum_{\text{f.p. } p}\ \prod_{\text{weights } w_i \text{ of } g \text{ action on } TM|_p}\frac{1}{w_i} \tag{4.89}$$

$$= \sum_k \Lambda^{4Nk}\int_{M_{N,k}} e^{\omega+H} \tag{4.90}$$

$$= \sum_k \Lambda^{4Nk}\int_{M_{N,k}} dxd\psi e^{\omega+H} \tag{4.91}$$

In the second line we used the Duistermaat-Heckmann theorem. Recalling that $M_{N,k} = \mu^{-1}\iota(\zeta^i)/U(k)$, we can enlarge the domain of integration to the entire $\mu^{-1}\iota(\zeta^i)$:

$$\int_{M_{N,k}} dxd\psi e^{\omega+H} = \int_{\mu^{-1}\iota(\zeta^i)} dxd\psi \int \frac{d\phi}{\text{vol}U(k)} d\bar{\phi}d\eta\, e^{+(w+H_\phi)+H} \tag{4.92}$$

Here ϕ^A and $\bar{\phi}^A$ takes value in the Lie algebra $u(k)$, V_A is the vector field of actions of $U(k)$ on $\mu^{-1}\iota(\zeta)$, and H_ϕ is the Hamiltonian generating $V_A\phi^A$. That this equality holds can be seen from the expansion

$$D(g_{\mu\nu}\psi^\mu V^\nu_A\bar{\phi}^A) = g_{\mu\nu}\phi^A V^\mu_A V^\nu_B\bar{\phi}^B + g_{\mu\nu}\psi^\mu V^\nu_A\eta^A. \tag{4.93}$$

The first term quadratic in ϕ and $\bar{\phi}$ can be trivially integrated out. The second term serves to eliminate extra fermions ψ along the symmetry direction $V^\nu_A\eta^A$.

Secondly, let us express the integration over the inverse image of the moment map μ using the delta functions. By further rewriting the delta functions using Fourier transformation, we get the representation for the equivariant index as an integral over all of $\mathbb{X}$:

$$\text{RHS of (4.92)} = \int \frac{d\phi}{\text{vol}U(k)} \int_{\mathbb{X}} dx d\psi \int d\bar{\phi} d\eta d\vec{\chi} d\vec{H} e^{D(\vec{\chi}\cdot(\vec{\mu}-\vec{\zeta}))+D(g\psi V\bar{\phi})+\omega+H_\phi+H} \tag{4.94}$$

where $\vec{\chi}$ are triplets of fermions taking value in $u(k)$ and $\vec{H}$ is their superpartners. This equation can be proved by inspecting that the first term of

$$D(\vec{\chi}\cdot(\vec{\mu}-\vec{\zeta})) = \vec{H}\cdot(\vec{\mu}-\vec{\zeta}) + \vec{\chi}\cdot D\vec{\mu} \tag{4.95}$$

gives the Fourier transformation of the delta functions and the second term provides the associated Fadeev–Popov determinant. We rewrite the result (4.94) a little by decomposing various vectors as $\vec{a} = (a_{\mathbb{C}}, a_{\mathbb{R}})$. Then, the quartet $\bar{\phi}, \eta, \chi_{\mathbb{R}}, H_{\mathbb{R}}$ can be trivially integrated out by the introduction of the mass term $tD(\chi_{\mathbb{R}}\bar{\phi})$ with t very large.

Inspecting the last expression, we see that when $\zeta_{\mathbb{C}} = 0$, the Duistermaat-Heckmann theorem can be applied to the integration over the supermanifold spanned by $\mathbb{X}$ and $\chi_{\mathbb{C}}$. As the supermanifold is a vector space and the action of the $U(k)$ and g are linear, the only fixed point is at the origin. Hence the contribution from the origin can be readily evaluated with the result

$$\text{RHS of (4.94)} = \int_{u(k)} d\phi \frac{\det_{\chi_{\mathbb{C}}}(\phi+J)}{\det_{\mathbb{X}}(\phi+J)} \tag{4.96}$$

Finally, noticing the integrand is invariant under the conjugation by $U(k)$ acting on ϕ, we can reduce the region of integration onto the Cartan subalgebra $\mathbb{R}^k$. The Vandermonde determinant appears in the integrand to ensure the size of the orbit passing through a point in $\mathbb{R}^k$. Combining all these consideration, we get

$$Z^{4.\text{dim}}_{\text{Nekrasov}} = \sum_k \Lambda^{4Nk} \int \prod_I \frac{\epsilon_1+\epsilon_2}{\epsilon_1\epsilon_2} \frac{d\phi_I}{P(\phi_I)P(\phi_I+\epsilon_1+\epsilon_2)} \prod_{I\neq J} \frac{\phi_{IJ}(\phi_{IJ}+\epsilon_1+\epsilon_2)}{(\phi_{IJ}+\epsilon_1)(\phi_{ij}+\epsilon_2)}, \tag{4.97}$$

where we introduced the notation $\phi_{IJ} \equiv \phi_I - \phi_J$. We can evaluate the integral using the generalization of Cauchy's integration formula, and it reproduces the fixed points and the residues described in section 4.4.2. However, we follow another path from here, inspired by an interpretation of the result (4.97) as a grand canonical ensemble of particles, with position ϕ_i, are interacting with each other inside a external potential. We will see that the Seiberg-Witten curves arise naturally from the consideration of the classical limit in the expression (4.97).

4.6 Dynamics of eigenvalues

This section reviews the solution of the dynamics of the particle system described by the equation (4.97). We follow closely the original article[14]. First, let us rewrite the equation (4.97) using the eigenvalue density

$$\rho(\phi) = \frac{1}{\epsilon}\sum_i \delta(\phi-\phi_i). \tag{4.98}$$

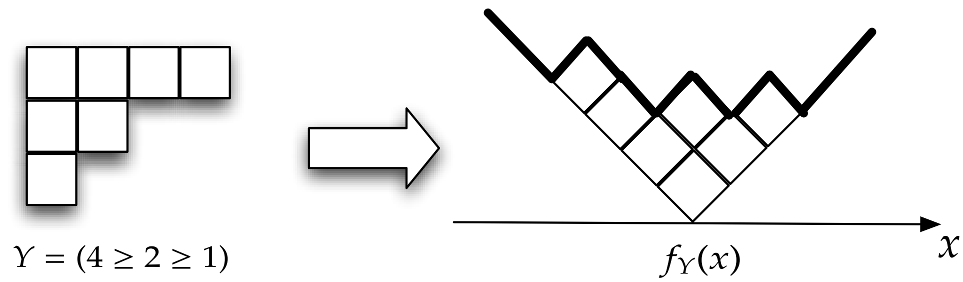


Figure 4.4: Definition of $f_Y(x)$

In the limit $\epsilon \to 0$, the result is

$$Z^{\text{4dim}}_{\text{Nekrasov}} = \int [d\rho] \exp(-E_{\text{inst}}[\rho]/\epsilon^2) \tag{4.99}$$

where

$$E_{\text{inst}}[\rho] = \int dx dy \frac{\rho(x)\rho(y)}{(x-y)^2} + 2 \int dx \rho(x) \log\left(\frac{P(x)^2}{\Lambda^{2N}}\right). \tag{4.100}$$

This action describes a grand canonical ensemble of particles on a one-dimensional line interacting with themselves by a two-body potential

$$-\frac{1}{(\phi_1 - \phi_2)^2} \tag{4.101}$$

in an external potential

$$\log P(\phi)^2/\Lambda^{2N}. \tag{4.102}$$

The dynamical scale Λ corresponds to the chemical potential for an eigenvalue in this description.

As argued in section 4.2.1, the prepotential of the system in the absence of graviphoton correction is given by

$$\mathscr{F}_0 = -\lim_{\epsilon\to 0} \epsilon^2 \log Z^{\text{4dim}}_{\text{Nekrasov}}. \tag{4.103}$$

Combined with the equation (4.99), we see that $\mathscr{F}_0$ is equal to $E[\rho_{\text{min}}]$ for a density profile ρ_{min} which minimizes the energy. This can be solved by utilizing a Riemann surface as usual for such a saddle point equation.

Energy functional from the residues at fixed points

Before solving the minimization problem, let us check that the same energy functional can be obtained from the Nekrasov's formula (4.85) which uses N-tuples of Young tableaux.

Firstly, introduce a function $f_Y(x)$ for a Young tableau $Y = (y_1 \geq y_2 \geq y_3 \geq \cdots)$ by

$$f_Y(x) = |x| + \sum_i \left(|x - \epsilon k_i + \epsilon(i-1)| - |x - \epsilon k_i + \epsilon i| + |x + \epsilon i| - |x + \epsilon(i-1)|\right). \tag{4.104}$$

The function describes the shape of the Young tableau as depicted in figure 4.4. For an N-tuplet of Young tableaux $\vec{Y} = (Y_1, \ldots, Y_N)$, we define

$$f_{\vec{Y}}(x) = \sum f_{Y_i}(x - a_i). \tag{4.105}$$

Then, the contribution of the fixed point specified by $\vec{Y}$ can be rewritten as

$$Z_{\text{pert}} Z_{\vec{Y}} = \exp\left(-\frac{1}{4}\int dxdy f''(x) f''(y) \gamma_\epsilon(x-y)\right) \tag{4.106}$$

where we define the function γ_ϵ by

$$\gamma_\epsilon(x+\epsilon) + \gamma_\epsilon(x-\epsilon) - 2\gamma_\epsilon(x) = \log\frac{x}{\Lambda} \tag{4.107}$$

having asymptotic expansion

$$\gamma_\epsilon(x) = \epsilon^{-2}\gamma_0(x) + \gamma_1(x) + \epsilon^2\gamma_2(x) + \cdots. \tag{4.108}$$

We further defined

$$Z_{\text{pert}} = \exp\left(\sum_{k,l}\gamma_\epsilon(a_l - a_n)\right). \tag{4.109}$$

Calculating the limit $\epsilon \to 0$, we can check

$$\epsilon^2 \log(Z_{\vec{Y}}) = E_{\text{inst}}[\rho] \tag{4.110}$$

after the identification $f(x) = \rho(x) + \sum |x - a_l|$. We define for convenience

$$E[f] \equiv \lim_{\epsilon\to 0}\epsilon^2 \log(Z_{\text{pert}} Z_{\vec{Y}}) \tag{4.111}$$

$$= E_{\text{inst}}[\rho] + \frac{1}{2}\sum (a_k - a_l)^2 \log\left(\left(\frac{a_l - a_k}{\Lambda}\right) - \frac{3}{2}\right). \tag{4.112}$$

As the additional term is independent of ρ, we can consider the minimization problem for $E[f]$ rather than $E_{\text{inst}}[\rho]$.

Solution of the minimization problem

We found in the previous sections that the instanton calculation of prepotential can be reduced to the minimization of the functional $E[f]$. $E[f]$ can be expressed as

$$E[f] = -\frac{1}{8}\int dxdy f''(x) f''(y)(x-y)^2\left(\log\left(\frac{x-y}{\Lambda}\right) - \frac{3}{2}\right) \tag{4.113}$$

where $f(x) = \rho(x) + \sum |x - a_l|$ is the shape of Young tableaux. We show below that

$$\mathscr{F}_{SW}(a) = \text{Min}_f E[f], \tag{4.114}$$

that is, the prepotential obtained by minimization is equal to the prepotential obtained from the Seiberg-Witten curve.

For convenience, we first Legendre-transform E with respect to variables a and define

$$S = E - \sum_l \xi_l a_l. \tag{4.115}$$

S can be defined as a functional of f by introducing a function

$$\xi(x) = \begin{cases} \xi_l & x \text{ is near } a_l \\ \text{smoothly interpolating} & \text{otherwise} \end{cases} \tag{4.116}$$

since

$$\sum_l \xi_l a_l = \int \xi(x) f''(x) dx. \tag{4.117}$$

We need to use basic relations

$$\int x|x-a|'' dx = a \tag{4.118}$$

$$\int x\rho(x)'' dx = 0 \tag{4.119}$$

to show this. Thus, after the Legendre transformation, what to minimize is the functional

$$S[f] = -\frac{1}{8}\int dxdy f''(x) f''(y)(x-y)^2 \left(\log\left(\frac{x-y}{\Lambda}\right) - \frac{3}{2}\right) + \frac{1}{2}\int \xi(x) f''(x) dx \tag{4.120}$$

We want to show that the value at the minimum is equal to

$$\mathscr{F}^D_{SW}(\xi) = \mathrm{Min}_f S[f] \tag{4.121}$$

under the identification

$$a^l_D = \xi_l, \qquad a_l = \frac{\partial \mathscr{F}^D_{SW}}{\partial \xi_l}. \tag{4.122}$$

Let us take the variation of S with respect to f. It results in equation

$$\int dy (y-x)\left(\log\left(\frac{|x-y|}{\Lambda}\right) - 1\right) f''(y) = \xi'(x). \tag{4.123}$$

Denote the left hand side by $g(x)$ and consider

$$\varphi(x) = f'(x) + \frac{g'(x)}{\pi i}. \tag{4.124}$$

We can depict the function $\varphi(x)$ as in figure 4.5 from the definition of $\xi(x)$. Furthermore, by differentiating the both sides of equation (4.124) by x, we obtain

$$\varphi(x)' = \mathrm{Im}\varphi(x)' + i\int \frac{dy}{x-y}\mathrm{Im}\varphi'(y). \tag{4.125}$$

Notice that the right hand side is none other than the Hilbert transform of the real function $f''(x)$, hence the relation is automatically satisfied if and only if $\varphi(x)$ can be extended holomorphically to $z \in \mathbb{H}$ the upper half plane and $\varphi(x)$ for $x \in \mathbb{R}$ is the boundary value of

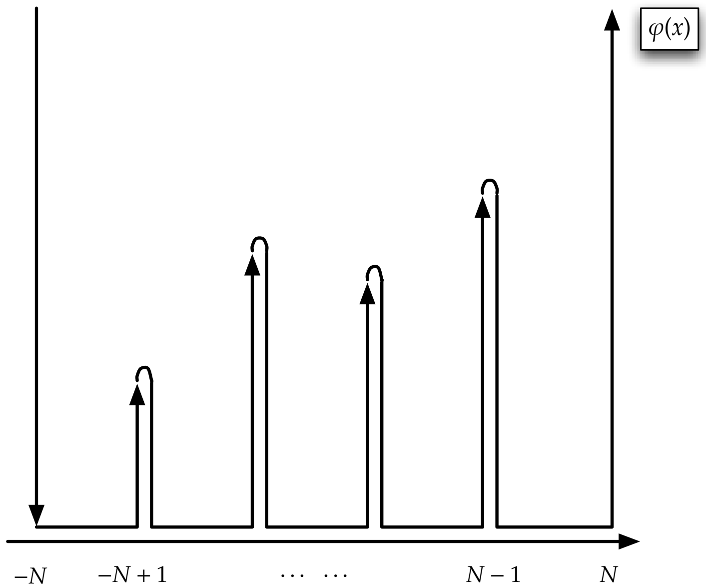


Figure 4.5: function $\varphi(x)$

$\varphi(z)$. The behavior of boundary value of $\varphi(z)$ described in figure 4.5 and the fact that it is holomorphic is sufficient to determine $\varphi(z)$ completely. Firstly, any holomorphic function with the behavior depicted in figure 4.5 can be expressed using a polynomial $Q(x)$ and the solution to the equation

$$w + \frac{1}{w} = \frac{Q(z)}{\Lambda^N} \tag{4.126}$$

as

$$\varphi(z) = \frac{2}{\pi i} \log w. \tag{4.127}$$

This is the same Seiberg-Witten curve we encountered before. To make this look more like that in equation (3.56), we need to change variables to $y = w + Q(x)$. We have

$$y^2 = Q(x)^2 + \Lambda^{2N}. \tag{4.128}$$

The mapping among z, w, φ as well as its value for $z \in \mathbb{R}$ is described in figure 4.6.

Then, ξ_l can be read off from ϕ using the relation

$$\xi_l - \xi_{l-1} = \int_{a_{l-1}^+}^{a_l^-} (-\pi)\mathrm{Im}\varphi dx = \oint_{B_l - B_{l-1}} x\frac{dw}{w}. \tag{4.129}$$

The derivative of $\mathrm{Min}_f S[f]$ with respect to ξ_l is

$$\frac{\partial}{\partial \xi_l}\mathrm{Min}_f S[f] = \frac{1}{2}\int_{a_l^-}^{a_l^+} x f''(x) = \frac{1}{2\pi i}\oint_{A_l} z\frac{dw}{w}. \tag{4.130}$$

Since

$$z\frac{dw}{w} \tag{4.131}$$

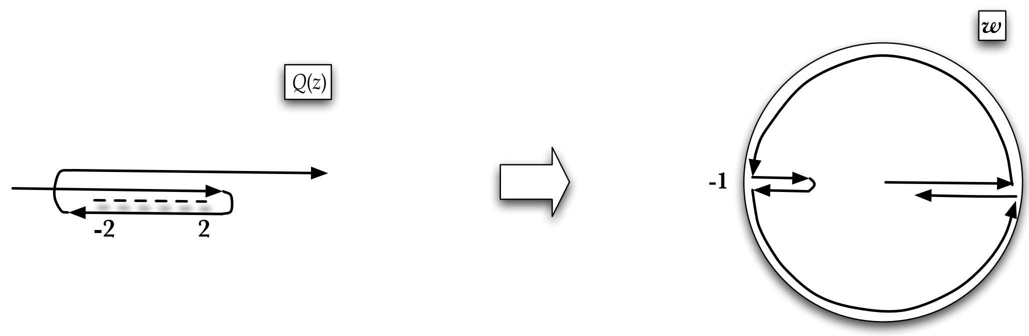


Figure 4.6: Mappings

is none other than the Seiberg-Witten prepotential, we have shown that

$$\mathscr{F}^{D}_{SW}(\xi_I) = \mathrm{Min}_f S[f]. \tag{4.132}$$

This is the desired result.

4.7 Effect of the five-dimensional Chern-Simons terms

This part is based on my paper [65].

4.7.1 Relation to the topological string amplitudes

Recently we saw a tremendous achievement in the calculation of all genus topological string amplitudes on local toric Calabi-Yau threefolds. The development was based on the observation that the toric Calabi-Yau and the worldsheet configuration in it can be cut using pairs of branes and anti-branes to $\mathbb{C}^3$. Closed string instantons wrapping holomorphic cycles can be cut into open string instantons. Then the amplitudes for open string worldsheet in $\mathbb{C}^3$ is calculated using the geometric transition and its relation to the three dimensional Chern-Simons theory. The method is called the topological vertex[66]. For a detailed account, we recommend the reader to consult the master thesis by R. Nobuyama[67] in Japanese.

As is argued almost ten years ago in Antoniadis *et al.*and in Bershadsky *et. al.*[68, 69], the topological string amplitude on a Calabi-Yau M can be physically interpreted as the graviphoton-corrected prepotential of the type IIA string theory compactified on M. Here the string coupling in the topological string side should be identified with the square of the field strength of the graviphoton. Using a non-compact Calabi-Yau, four-dimensional gravity can be decoupled from the gauge theory. However, as the type IIA string theory inherently contains eleven-dimensional phenomena, just compactifying the system on a local toric Calabi-Yau with finite Kähler parameters does not produce genuine four-dimensional theory. Rather it gives a certain five-dimensional theory, obtained by compactifying M-theory on local toric Calabi-Yau M. In order to reproduce the four dimensional result, we need to take some controlled limit of various Kähler parameters of M.

From extensive works of Vafa and collaborators on the geometric engineering of $\mathcal{N}=2$ gauge theory in four dimensions, we know the toric Calabi-Yau X that should be used to produce pure four-dimensional $U(N)$ $\mathcal{N}=2$ supersymmetric gauge theory[70, 71]. There are several of them, labeled by a integral parameter m. We denote them X_N^m. In the original works of Vafa et.al., the authors showed that taking suitable limits the topological amplitudes reproduce the celebrated prepotentials of Seiberg and Witten.

Now, we are in a rare situation that using the method of topological vertex, we can obtain the all-genus topological A-model amplitudes for the local Calabi-Yau manifolds X_N^m with Kähler parameters being finite, and compare them to the exact result of five dimensional gauge theory. On the topological string side, the calculation was originally carried out by Iqbal and Kashani-Poor with some simplifying mathematical hypothesis. Later Eguchi and Kanno [60, 61] proved the hypothesis and extended the calculation to more general toric Calabi-Yau manifolds. On the gauge theory side, we have the conjecture made by Nekrasov. His calculation was carefully reviewed in the previous sections.

The outcome was somewhat surprising. Of the topological amplitudes for X_N^m , only one of them, that for X_N^0, coincided with the result by Nekrasov. This naturally leads to the question: how do the five dimensional theories obtained by compactifying M-theory on X_N^m differ from each other, and how can we reproduce the topological A-model amplitudes in the gauge theory side, *à la* Nekrasov?

The first part of the question is, in fact, already answered in the literature. The answer to the second part is our original contribution. After reviewing the answer to the first part, we will see that by slightly extending the analysis by Nekrasov we can obtain the graviphoton-corrected prepotential of five-dimensional gauge theory corresponding to X_N^m. The results exactly match with those obtained by Iqbal and Kashani-Poor.

4.7.2 Triple intersection and the Chern-Simons terms

Firstly we recall the generic structure of five dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theory[72]. This will tell us how the M-theory compactification gives us different gauge theories with the same gauge group.

Since a five dimensional $\mathcal{N}=1$ gauge theory yield a four dimensional $\mathcal{N}=2$ gauge theory by a simple dimensional reduction, the five dimensional $U(1)^n$ theory can also be summarized by a holomorphic prepotential $\mathscr{F}$. We saw in section 3.1.1 that this leads to the following scalar dependent gauge coupling

$$\propto \frac{\partial^2 \mathscr{F}}{\partial a_i \partial a_j} F_i^+ \wedge F_j^+ \tag{4.133}$$

This in particular includes the scalar dependent θ-term

$$\left(\mathrm{Re}\frac{\partial^2 \mathscr{F}}{\partial a_i \partial a_j}\right) F_i \wedge F_j. \tag{4.134}$$

In five dimension, however, part of the scalar comes from a component of gauge field along the fifth direction. Thus the term above is of the form

$$f(A_{k,5}) F_i \wedge F_j, \tag{4.135}$$

thus it is seldom gauge invariant since it contains the gauge potentials A directly. Only when $f(A_k)$ is linear in the gauge potential it becomes gauge invariant, since the term forms a five dimensional Chern-Simons term in such cases. This means that the prepotential is a polynomial of degree up to three, $\mathscr{F} = c_{ijk}a_i a_j a_k + \tau_{ij}a_i a_j$. This gives the Chern-Simons term

$$\int c_{ijk} A_i \wedge F_j \wedge F_k \tag{4.136}$$

for the five dimensional $\mathscr{N} = 1$ $U(1)^n$ gauge theories.

Secondly let us see how the coefficients c_{ijk} is determined from the geometric data, when the theory is realized by a M-theory compactification. When M-theory is 'compactified' on a Calabi-Yau manifold, it should yield a theory with eight supersymmetries. This is because the Calabi-Yau geometry breaks three fourths of original thirty-two supersymmetries. Furthermore, we need a non-compact manifold in order to decouple the field theory from the gravity. For this reason we put a quotation marks around the word 'compactified' above. In this setup, five dimensional vector fields A_i come from the three form field $C^{(3)}$ of the eleven dimensional supergravity reduced along two cycles C_i in the Calabi-Yau, $A_i = \int_{C_i} C^{(3)}$. The five dimensional Chern-Simons term (4.136) comes directly from the eleven dimensional Chern-Simons coupling

$$\int C^{(3)} \wedge (dC)^{(4)} \wedge (dC)^{(4)}. \tag{4.137}$$

Thus we see that the coefficient c_{ijk} is none other than the triple intersection of (the Poincaré duals of) two cycles C_i.

Calabi-Yau space can develop singularities when the Kähler parameters are suitably chosen. When the Calabi-Yau space develop an ADE singularity through the collapse of two cycles, there appears enhanced non-abelian gauge symmetry corresponding to the ADE type of the singularity. The W-bosons corresponding to the roots of the gauge group is provided by the M2-brane wrapped around the collapsed cycles. In such cases, the Chern-Simons coupling (4.136) should be likewise enhanced to the non-abelian version $CS(A, F)$ which is defined through the decent construction

$$dCS(A, F) = \mathrm{tr}(F \wedge F \wedge F) \tag{4.138}$$

where F is the non-abelian field strength. Moreover, Intriligator *et. al.* [73] studied the geometry of general Calabi-Yau manifolds which give rise five dimensional $SU(N)$ theory and showed that the triple intersection is determined up to the coefficient of this non-abelian Chern-Simons terms.

Iqbal and Kashani-Poor studied M-theory compactification on local toric Calabi-Yau manifolds X_N^m. We collect here relevant facts on those manifolds without proof. For more detailed account please refer their article [71].

X_N^m is a fibration of A_{N-1} singularity over the base $\mathbb{CP}^1$. It contains a sequence of compact divisors

$$S_i(m) \in \{\mathbb{F}_{m+2-N}, \mathbb{F}_{m+4-N}, \ldots, \mathbb{F}_{m+N-2}\} \tag{4.139}$$

Here $\mathbb{F}_n$ denotes the Hirzebruch surface. The Hirzebruch surface $\mathbb{F}_m$ is a $\mathbb{CP}^1$ fibration over the $\mathbb{CP}^1$ with the intersection pairing

$$\mathbb{CP}^1_{\text{base}} \cdot \mathbb{CP}^1_{\text{base}} = m, \tag{4.140}$$
$$\mathbb{CP}^1_{\text{base}} \cdot \mathbb{CP}^1_{\text{fiber}} = 1, \tag{4.141}$$
$$\mathbb{CP}^1_{\text{fiber}} \cdot \mathbb{CP}^1_{\text{fiber}} = 0. \tag{4.142}$$

An A_{N-1} singularity contains at the tip $N-1$ $\mathbb{CP}^1$'s $C_1, \ldots, C_{N-1}$ with intersection pairing $C_i \cdot C_{i+1} = 1$. The divisor S_i of X_N^m is the fibration of C_i over the base $\mathbb{CP}^1$.

The prepotential is given by the formula

$$\mathscr{F}_m = \frac{1}{2}\sum_{i,j} |a_i - a_j|^3 + m \sum a_i^3 \tag{4.143}$$

where

$$\mathscr{F} = \sum_{i,j,k} (a_{i+1} + \cdots + a_N)(a_{j+1} + \cdots + a_N)(a_{k+1} + \cdots + a_N)(S_i \cdot S_j \cdot S_k). \tag{4.144}$$

From these expressions we see that the label m of X_N^m is exactly proportional to the magnitude of the five dimensional Chern-Simons term.

4.7.3 Calculation *à la* Nekrasov

Let us now investigate what are the effect of five-dimensional Chern-Simons terms on the calculation *à la* Nekrasov. The setup is essentially the same as in section 4.4. We put the theory on the Ω background and we encode the moduli of the theory by introducing Wilson lines along the fifth direction. The determination of the Q-fixed configuration is also the same; it localizes the calculation to that on trajectories in the moduli of ASD instantons. The calculation is now reduced to that of supersymmetric quantum mechanics on the ASD instanton moduli space. The difference lies in the Hamiltonian of the quantum mechanical system. The Lagrangian of the quantum mechanical system is obtained by substituting the gauge field in the five-dimensional action by corresponding anti-self-dual configurations specified by the trajectory in the moduli space. The Yang-Mills action gives the kinetic term for the point particle moving on the ASD moduli, and the Chern-Simons term gives a phase depending on the trajectory:

$$e^{m\int CS(A,F)} = e^{im\int dx^\mu A_\mu} \tag{4.145}$$

The point particle is now coupled with an external vector potential on the instanton moduli. Therefore, the exact partition function of the theory put on the Ω background is

$$Z = \sum_k q^{Nk} \mathrm{Ind}_g(M_{N,k}, \mathscr{L}^{\otimes m}) \tag{4.146}$$

where $\mathscr{L}$ is the complex line bundle determined by the phase from the Chern-Simons term with coefficient one. In view of the fixed point theorem reviewed in section 2.4, the precise understanding of the line bundle $\mathscr{L}$ gives the answer.

The line bundle $\mathscr{L}$ has, in fact, long been known to physicists. It is the so-called determinant bundle $\mathrm{Det}\not{D}$. The determinant line bundle is defined on the space of connections $\mathscr{A}/\mathscr{G}$ and the fiber at a configuration A is defined by $(\mathrm{detKer}\not{D}_A)^* \otimes \mathrm{detKer}\not{D}_A^\dagger$ where

$D_A : \Gamma(S^- \otimes E) \to \Gamma(S^+ \otimes E)$ is the chiral Dirac operator coupled to the connection A. When the base space is restricted on the ASD moduli space, it can be simplified to $(\text{detKer}\not{D}_A)^*$ because we know that there are no positive chirality zero modes. Close relation between the determinant line bundle and the Chern-Simons terms is known from the work of Alvarez-Gaumé and Ginsparg [74] on the geometric reinterpretation of non-abelian anomalies. For the direct construction of the hermitian metric and connections on the determinant line bundle we refer the reader to the exposition of Bismut and Freed[75]. In reality we need to blow up the small instanton singularity in the ASD moduli using spacetime non-commutativity. Hence we need the non-commutative extension of all these familiar facts. Fortunately every detail we need is already worked out by various groups following the seminal work of Seiberg and Witten on noncommutativity[30]. We refer the reader the work [76, 77] for noncommutative extension of the relation of non-abelian anomalies and the index theorem in six dimesions, and the work [78] for the study of the Dirac zero modes in the non-commutative instanton background.

Now that we have clear understanding on the nature and the structure of the line bundle $\mathscr{L}$, we can complete the calculation. Using the Atiyah-Bott-Lefschetz fixed point formula, or the localization to the supersymmetric quantum mechanics if we prefer the physics language, we can rewrite the equivariant index as

$$\text{Ind}_g(M_{N,k}, \mathscr{L}^{\otimes m}) = \sum_{\text{f.p. } p} e^{imw} \prod_{g_i} \frac{1}{\sinh \frac{\beta}{2} g_i} \tag{4.147}$$

where g_i denotes the eigenvalue of g on the tangent space $TM_{N,k}|_p$ and w denotes the eigenvalue of g action on the fiber $\mathscr{L}^{\otimes m}|_p$. As seen in the previous subsection, $\mathscr{L}$ is none other than the determinant line bundle of the Dirac operator coupled to the ASD connection. Recalling that generically there are no wrong chirality zero-modes in the ASD background, the fiber at p is the highest exterior power of the kernel of the Dirac operator. Thus, to determine the weight w, we have to determine the action of g on the Dirac zero-modes.

Fortunately, we know already the location of the fixed points on the instanton moduli space, and we know how to construct the Dirac zero-modes from the ADHM data. As seen in section 4.4.1, the ADHM data X_p itself is not invariant under the action of g, it maps X_p to a datum equivalent under $U(k)$ transformation ϕ:

$$gX_p = \phi(g)X_p. \tag{4.148}$$

We know from the analysis in section 1.2.2 that $U(k)$ acts on the $4k$ Dirac zero-modes on the k-instanton background in the fundamental representation. These arguments show that the action of g on the zero-modes can be traded to the action of $\phi(g)$. Moreover, the action of $\phi(g)$ is already determined in section 4.4.1 and the weight w can be readily computed

$$w = \sum_k \sum_{(i,j)\in Y_k} (a_k + \epsilon(i-j)) \equiv \sum_k \left(l_{Y_k} a_k + \kappa_k \right) \tag{4.149}$$

where $(Y_1, \ldots, Y_N)$ is the N-tuplets of Young tableaux specifying the non-commutative instantons invariant under the action of g. We denoted by l_i the number of boxes in Y_i and

$$\sum_{(i,j)\in Y_i} (i-j) \tag{4.150}$$

by κ_i.

Finally we get the partition function for the five-dimensional theory with non-abelian Chern-Simons term on the Ω-background:

$$Z = \sum_{Y_1,\dots,Y_N} \phi^{\sum_i l_i} e^{-m\beta\sum(l_i a_i + \kappa_i)} \prod_{l,n=1}^{N} \prod_{i,j=1}^{\infty} \frac{\sinh\frac{\beta}{2}(a_{ln} + \epsilon(y_{l,j} - y_{n,i} + j - i))}{\sinh\frac{\beta}{2}(a_{ln} + \epsilon(y_{l,j} - y_{n,i}))}. \tag{4.151}$$

We defined $a_{ij} = a_i - a_j$ for brevity.

4.7.4 Comparison with the topological A-model amplitudes

Let us compare what we have obtained *à la* Nekrasov against the topological A-model amplitudes for local toric Calabi-Yau manifolds X_N^m. Combining the equations (67,68,69,78) in the article by Iqbal and Kashani-Poor[71], the amplitude is

$$\begin{aligned} Z_{\text{topological}} = \sum_{Y_1,\dots,Y_N} & 2^{-2N\sum l_{Y_i}} (-)^{(N+m)\sum l_{Y_i}} q^{\frac{1}{2}\sum_{i=1}^{N}(N+m-2i)\kappa_i} Q_B^{\sum l_i} \times \\ & \prod_{i=1}^{\lfloor\frac{N+m-1}{2}\rfloor} Q_i^{(N+m-2i)(l_1+\dots+l_i)} \prod_{i=\lfloor\frac{N+m+1}{2}\rfloor}^{N-1} Q_i^{(2i-m-N)(l_{i+1}+\dots+l_N)} \prod_{i=1}^{N-1} Q_{b_i}^{-(N-i)(l_1+\dots+l_i)-i(l_{i+1}+\dots+l_N)} \\ & \times q^{-\frac{1}{2}\sum_{i=1}^{N}(N-2i)\kappa_i} \prod_{l,n=1}^{N} \prod_{i,j=1}^{\infty} \frac{\sinh\frac{\beta}{2}(a_{ln} + \epsilon(y_{l,j} - y_{n,i} + j - i))}{\sinh\frac{\beta}{2}(a_{ln} + \epsilon(y_{l,j} - y_{n,i}))} \end{aligned} \tag{4.152}$$

where Q_B and $Q_i = e^{-\beta(a_i - a_{i+1})}$ are respectively the exponential of the Kähler parameters of the base divisor and the divisors S_i. Define a_N by

$$e^{-\beta a_N} = -\left(Q_B \prod_{i=1}^{\lfloor\frac{N+m-1}{2}\rfloor} Q_i^{-i} \prod_{i=\lfloor\frac{N+m+1}{2}\rfloor}^{N-1} Q_i^{-(N-i+m)} \right)^{\frac{1}{m}}. \tag{4.153}$$

Then, after reshuffling the various factors with some effort, one finds that

$$Z_{\text{topological}} = \sum_{Y_1,\dots,Y_N} (-4)^{N\sum l_i} e^{-m\beta\sum(l_i a_i + \epsilon\kappa_i)} \prod_{l,n=1}^{N} \prod_{i,j=1}^{\infty} \frac{\sinh\frac{\beta}{2}(a_{ln} + \epsilon(y_{l,j} - y_{n,i} + j - i))}{\sinh\frac{\beta}{2}(a_{ln} + \epsilon(y_{l,j} - y_{n,i}))}. \tag{4.154}$$

Now we can see this exactly matches with the calculation of the gauge theory side *à la* Nekrasov. This is the desired result.

Chapter 5

Conclusion

In this master thesis, we reviewed the development in the last two years on the instanton calculation of prepotentials using localization. The calculation consisted of three main points. The first is the identification of the (graviphoton-corrected) prepotential with the logarithm of the partition function on the Ω background. The second ingredient is the rephrasing of the partition function as the equivariant indices of the instanton moduli. The third one is the detailed understanding of the instanton moduli and fixed points on it using the ADHM description. We saw that the resulting formula precisely matched with the prepotential obtained from the Seiberg-Witten curves. This presented a good consistency checks for both the strong coupling holomorphy calculation *à la* Seiberg and Witten and the weak coupling instanton calculation *à la* Nekrasov.

In hindsight, the mathematical machinery needed for the Nekrasov formulae was well prepared already twenty years ago. Hence, the first two preparatory chapters consisted of somewhat old materials. These works were done mainly by M. Atiyah and his collaborators. It is fun to imagine what would have happened if they calculated the equivariant index of the ADHM moduli spaces. They could have found the integrable structures in the K-theory of those spaces. Of course people would not do anything without a good motivation. However, the author personally thinks that the recent development tells us that there may be more physically interesting topics to be uncovered, which is hidden in the well-understood mathematical tools and machineries.

A few natural extension of these works come to our mind. One is the generalization to more general gauge groups and matter representations. Extension for the classical gauge groups with fundamental or adjoint hypermultiplets have essentially been done in the original works. The extension to the exceptional groups is a more challenging task. It will be interesting to study its relation with the Seiberg-Witten geometry for the exceptional groups. Another is the generalization of the geometry of the spacetime. In this review we treated the case of $\mathbb{R}^4$. In order to apply the localization theorem we need manifolds with two commuting isometries. If we further assumes the spacetime to be Kähler, it means that we should deal with toric two-folds. For $\mathbb{CP}^2$ and its one-point blowups, the analysis was done by Nakajima and Yoshioka [15] and they showed the WDVV equation from these consideration. The study of more generalization and its relation to the integrable structure seem very fruitful.

An exciting development concerning the Nekrasov's formula that we could not fully cover in the thesis is the relation to the topological strings. The point is that the five dimensional extension of the Nekrasov's formula and the all-genus partition function of the topological strings completely agree for cases which can be computed on both sides. It will be worth while to establish the equality for M-theory compactifications on general local toric Calabi-Yau manifolds. For that purpose, it will be necessary to understand the topological vertex directly in the four-dimensional gauge theory side, and to calculate them using the localization. From the works of Ooguri and Vafa [79], we know that the topological vertex should correspond to the $\mathcal{N} = 1$ gauge theory in four dimensions. Hence we must first extend the localization calculation to the $\mathcal{N} = 1$ gauge theory. This suggests strong relation with the Dijkgraaf-Vafa method[80, 81]. It will be very interesting to compare the approach of Nekrasov and that of Dijkgraaf and Vafa. Moreover, we need to understand the propagator of the topological Feynman rules in the four-dimensional gauge theory. This also is worth while to pursue. Finally, we would like to note that in the toplogical Feynman rules we construct closed string amplitudes from open string amplitudes. If this can be translated to physical strings naïvely, it should mean that we could construct a $\mathcal{N} = 2$ gauge theory by sewing several $\mathcal{N} = 1$ gauge theories. This sounds very intriguing and we hope that the study in these directions will yield a better understanding of the restoration of supersymmetry after brane-antibrane annihilation.

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