Matching higher symmetries across Intriligator-Seiberg duality

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with **Yasunori Lee** and **Kantaro Ohmori**

2108.05369

Strings in Seoul, KIAS, Sep. 3, 2021

With two fantastic collaborators...

Yasunori Lee

Kantaro Ohmori

We studied how higher symmetries

i.e. 1-form symmetries, 2-groups and their anomalies

match across the 4d $\mathcal{N}=1$ duality of Intriligator and Seiberg,

between $\mathfrak{so}(2n_c) \leftrightarrow \mathfrak{so}(2n_f - 2n_c + 4)$ with $2n_f$ flavors.

It took us almost two years of thinking on-and-off ...

which had many ups and downs (but mostly downs) ...

which I would like to recount, but no!

Let's proceed.

I will review

- 1-form symmetries and 2-groups
- Intriligator-Seiberg duality

and then

- describe how they are combined.

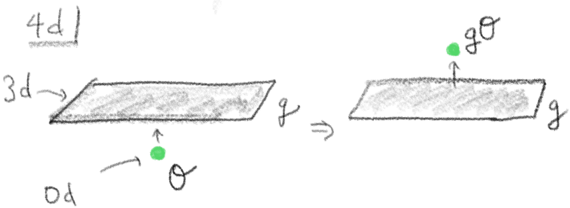
There would be a lot of overlaps with Sakura's talk earlier in this conference.

Higher symmetries

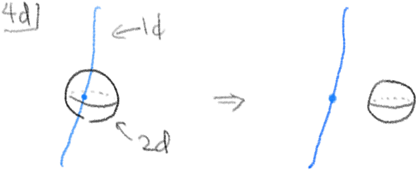
Symmetry $g \in G$ in 4d can be visualized as

an operator $\mathcal{O}$ crossing a 3d wall labeled by g.

Take $G = \mathbb{Z}_2$. If $\mathcal{O}$ is odd,

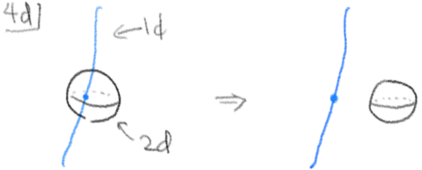


it gets multiplied by -1 when crossing the wall.

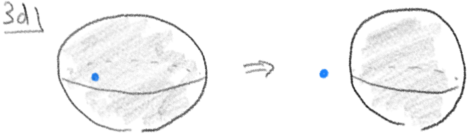


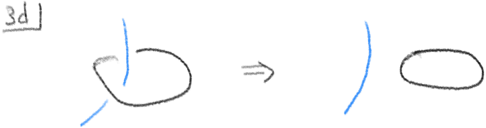
Can consider “symmetry” acting on a line operator $\boldsymbol{L}$,
rather than a point operator $\boldsymbol{\mathcal{O}}$.

Captured by a 1d world-line crossing a 2d wall.



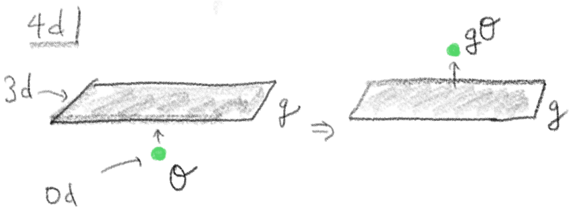
looks differently depending on how to project it:



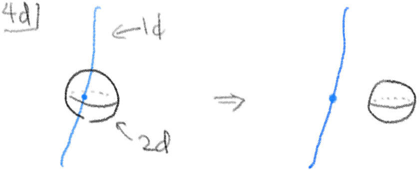


A p-form symmetry is a "symmetry" which acts on p-dim'l objects

0-form symmetry



1-form symmetry



[Gaiotto-Kapustin-Seiberg-Wilett, 1412.5148]

You know that 0-form symmetry groups can be extended:

$$\mathbf{0 \to \boldsymbol{H} \to \Gamma \to \boldsymbol{G} \to 0.}$$

1-form symmetry can also extend **0-form symmetry**:

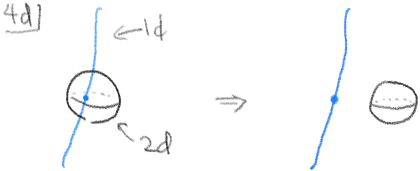
$$\mathbf{0 \to \boldsymbol{A}[1] \to \Gamma \to \boldsymbol{G} \to 0}$$

where $\mathbf{\Gamma}$ is now a mixture of 0-form and 1-form symmetry, often called a **2-group**.

The 4d gauge theory case was first found in [Hsin-Lam, 2007.05915]

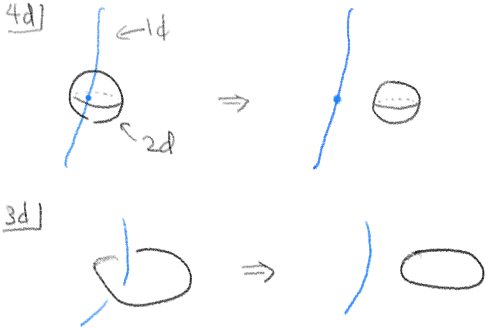
Enough with abstract non-sense!

Let's see some examples.



Maxwell = pure $\mathbf{SO(2)}$ gauge theory has

- Electric $\mathbb{Z}_2$ **1-form symmetry**:
  $(-1)^q$ when crossing a worldline of electric charge q,
- magnetic $\mathbb{Z}_2$ **1-form symmetry**:
  $(-1)^m$ when crossing a worldline of magnetic charge m



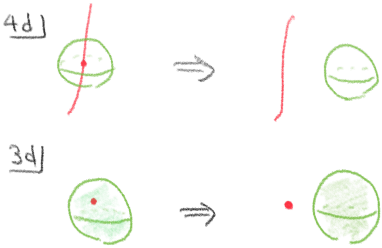
$(-1)^q$ when crossing a worldline of electric charge q.

$(-1)^q$ when crossing a worldline of electric charge q.

$$\exp(2\pi i q \int_C \vec{A} \cdot d\vec{x}) \mapsto \exp(2\pi i q \int_{C'} \vec{A} \cdot d\vec{x})(-1)^q$$

This means that the black wall realizing the electric $\mathbb{Z}_2$ 1-symmetry has **half the flux** of the magnetic quantum

$$\vec{B} = \oint \vec{A} \cdot d\vec{x} = \int_C \vec{A} \cdot d\vec{x} - \int_{C'} \vec{A} \cdot d\vec{x} = \pm\frac{1}{2}$$



$(-1)^m$ when crossing a worldline of magnetic charge m.

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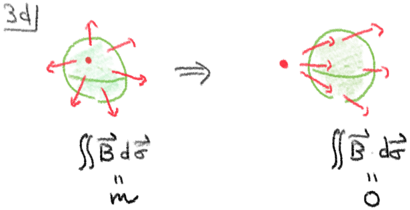


This means that the green wall realizing the magnetic $\mathbb{Z}_2$ **1-symmetry** has the factor

$$\exp(\pi i \iint \vec{B} \cdot d\vec{\sigma})$$

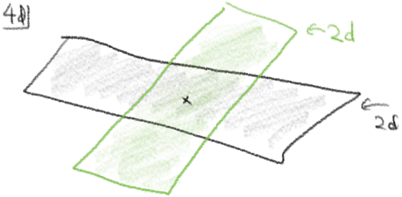
Wall for electric $\mathbb{Z}_2$ **1-symmetry**

has $\vec{B} = \pm\frac{1}{2}$ around it

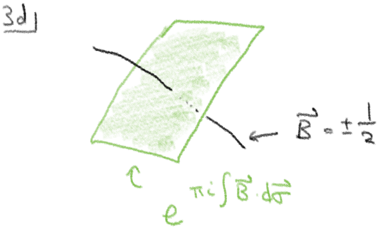
Wall for magnetic $\mathbb{Z}_2$ **1-symmetry**

$$\exp(\pi i \iint \vec{B} \cdot d\vec{\sigma})$$

Problematic if both walls are inserted at the same time, since two 2d surfaces intersect at points in 4d.



If depicted in one lower dimension,



You can't tell if the phase is which of

$$e^{\pm\pi i/2} = \quad +i\,? \quad \text{or} \quad -i\,?$$

This is a $\{\pm 1\}$-valued mixed anomaly
between electric and magnetic $\mathbb{Z}_2$ 1-form symmetries.

Let us next consider pure $\mathbf{SO(2n)}$ gauge theory, which also has

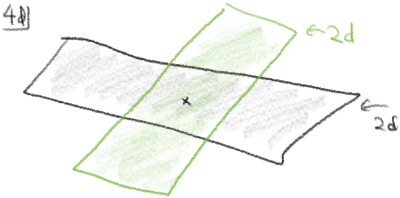
- **Electric $\mathbb{Z}_2$ 1-form symmetry:**

  A Wilson line in rep. $\boldsymbol{R}$ of $\mathbf{SO(2n)}$ has charge $\boldsymbol{q = 0, 1}$ when $\mathbf{-1 \in SO(2n)}$ acts as $\mathbf{(-1)^{q}}$

- **Magnetic $\mathbb{Z}_2$ 1-form symmetry:**

  't Hooft lines carry $\mathbb{Z}_2$ charge given by $\int_{S^2} \boldsymbol{w_2}$, where $\boldsymbol{w_2}$ is the Stiefel-Whitney class of $\mathbf{SO(2n)}$ controlling whether the bundle lifts to $\mathbf{Spin(2n)}$.

Also for the pure $\mathbf{SO(2n)}$ gauge theory, there can be a mixed anomaly:



where the partition function is ambiguous by a sign $(-1)^n$.

This can be found by breaking $\mathbf{SO(2n)} \to \mathbf{SO(2)}^n$.

Each $\mathbf{SO(2)}$ contributes by $(-1) \to (-1)^n$ in total.

$\mathbf{SO(2n)}$ theory has magnetic $\mathbb{Z}_2$ 1-form symmetry,
which measures $m = \int_{S^2} w_2$.

Gauge this magnetic $\mathbb{Z}_2$ 1-form symmetry

$\to$ charged lines with $m \neq 0$ removed

$\to$ all configurations liftable to $\mathbf{Spin(2n)}$

$\to$ becomes the $\mathbf{Spin(2n)}$ theory.

Pure $\mathbf{Spin}(2n)$ theory has electric 1-form "center symmetry", which is given by:

$$\text{center of } \mathbf{Spin}(2n) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & (n\text{: even}) \\ \mathbb{Z}_4 & (n\text{: odd}) \end{cases}$$

For pure $\mathbf{Spin}(2n)$ and $\mathbf{SO}(2n)$ theories, we have

	$\mathbf{Spin}(2n)$		$\mathbf{SO}(2n)$
n: even	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anomaly	gauge $\mathbb{Z}_2$ $\longrightarrow$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anomaly
n: odd	$\mathbb{Z}_2 \subset \mathbb{Z}_4$	$\longleftarrow$ gauge $\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ with anomaly

where $\mathbb{Z}_4/\mathbb{Z}_2 = \mathbb{Z}_2$, i.e.

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0.$$

Note that these are formal features independent of specific models.

[YT, 1712.09542]

(Every symmetry on this slide is 1-form.)

$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anomaly	gauge $\mathbb{Z}_2$ $\longrightarrow$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anomaly
$\mathbb{Z}_2 \subset \mathbb{Z}_4$	$\longleftarrow$ gauge $\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ with anomaly

where $\mathbb{Z}_4/\mathbb{Z}_2 = \mathbb{Z}_2$, i.e.

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Note that these are formal features independent of specific models.

[YT, 1712.09542]

(Every symmetry on this slide is 1-form.)

I think we got some good ideas on 1-form symmetries.

Let us move on to 2-groups.

We consider $\mathbf{Spin}(2n_c)$ gauge theory
with $2n_f$ flavors in the **vector** representation.

It has $\mathfrak{su}(2n_f)$ flavor symmetry.

What is the 1-form symmetry?

The center of $\mathbf{Spin}(2n_c)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$.

In particular,

$$\text{spinor} \otimes \text{spinor} = \text{vector}$$

in the latter.

But the Wilson line in the vector representation can now be screened by dynamical particles, i.e.

$$\text{vector} \xrightarrow{\text{screen}} \text{trivial}$$

Only the $\mathbb{Z}_2$ quotient group survives,
generated by Wilson lines in the spinor representation.

$\longrightarrow$ only $\mathbb{Z}_2$ 1-form symmetry remains.

But wait!

The dynamical particle used in screening is in the

vector $\otimes$ **fundamental**

representation of $\mathbf{Spin(2n_c)} \times \mathbf{SU(2n_f)}$.

$\mathbf{-1} \in \mathbf{SU(2n_f)}$ acts nontrivially on it.

For $\mathbf{Spin}(2n_c)$ with n_c even and with $2n_f$ flavors:

$$\text{(gauge spinor)}^{\otimes 2} = \text{gauge singlet,}$$

For $\mathbf{Spin}(2n_c)$ with n_c odd and with $2n_f$ flavors:

$$\text{(gauge spinor)}^{\otimes 2} = \text{gauge vector} \xrightarrow{\text{screen}} \textbf{flavor fundamental}$$

When n_c is odd,
1-form symmetry and 0-form flavor symmetry are intrinsically mixed.

Known as a 2-group symmetry.

Formalizing it mathematically is a bit tiresome,
but the physics content is basically what I just described.

In other words:

$\mathbf{SU}(2n_f)$ 0-form flavor symmetry

$\xrightarrow{\text{extract}}$ effective **flavor $\mathbb{Z}_2$ 1-form** symmetry under which
favor fundamental Wilson lines are charged

Then:

$$\mathbf{0} \to \textbf{electric } \mathbb{Z}_2 \textbf{ 1-form} \text{ symmetry} \to \left\{ \begin{array}{ll} \mathbb{Z}_2 \times \mathbb{Z}_2 & (n_c\text{: even}) \\ \mathbb{Z}_4 & (n_c\text{: odd}) \end{array} \right\} \to \textbf{flavor } \mathbb{Z}_2 \textbf{ 1-form} \text{ symmetry} \to \mathbf{0}$$

For $\mathbf{Spin}(2n_c)$ and $\mathbf{SO}(2n_c)$ theories with $2n_f$ **scalar** flavors,

	$\mathbf{Spin}(2n)$		$\mathbf{SO}(2n)$
n: even	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anomaly	gauge $\mathbb{Z}_2$ $\longrightarrow$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anomaly
n: odd	$\mathbb{Z}_2 \subset$ "$\mathbb{Z}_4$"	$\longleftarrow$ gauge $\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ with anomaly

where

$\mathbb{Z}_2$:	electric	1-form
$\mathbb{Z}_2$:	magnetic	1-form
$\mathbb{Z}_2$:	"flavor"	1-form for $\{\pm 1\} \subset \mathbf{SU}(2n_f)$

and

$$\text{“}\mathbb{Z}_4\text{”}/\mathbb{Z}_2 = \mathbb{Z}_2.$$

(Note that fermions will change many things,
due to potential anomalies.)

Note that this is not very different from the case of pure $\mathbf{Spin}(2n)$ and $\mathbf{SO}(2n)$ theories:

	$\mathbf{Spin}(2n)$		$\mathbf{SO}(2n)$
n: even	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anomaly	gauge $\mathbb{Z}_2$ $\longrightarrow$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anomaly
n: odd	$\mathbb{Z}_2 \subset \mathbb{Z}_4$	$\longleftarrow$ gauge $\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ with anomaly

where $\mathbb{Z}_4/\mathbb{Z}_2 = \mathbb{Z}_2$.

When there are $2n_f$ flavors, we just re-interpret the blue $\mathbb{Z}_2$ part as coming from $\{\pm 1\} \subset \mathbf{SU}(2n_f)$.

Intriligator-Seiberg duality

[Intriligator-Seiberg hep-th/9503179] found the duality

$$\begin{array}{lcl} \text{4d } \mathcal{N}{=}1 & \mathfrak{so}(2n_c) & \text{with } 2n_f \text{ flavors} \\ & \updownarrow & \\ \text{4d } \mathcal{N}{=}1 & \mathfrak{so}(2n_f - 2n_c + 4) & \text{with } 2n_f \text{ flavors} \end{array}$$

Many checks: 0-form symmetries, anomaly polynomial, SCI ...

How about the global form of the gauge group?

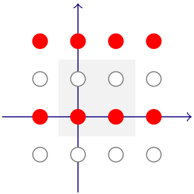
An early work [Strassler hep-th/9709081] suggested that **SO** $\leftrightarrow$ **Spin**, but did not quite uncover the whole story.

Strassler added massive spinor flavors on the electric side and studied how it affects the magnetic side.

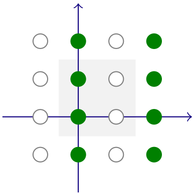
If we take the infinite mass limit, we can re-interpret his analysis as a study of spinor Wilson line operators.

A streamlined analysis was given in [Aharony-Seiberg-YT 1305.0318].

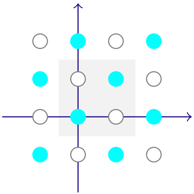
There are in fact three types of $\mathfrak{so}$ QCD:



Spin
has Wilson line



$\mathbf{SO}_{+}$
has 't Hooft line



$\mathbf{SO}_{-}$
has dyonic line

Under the Intriligator-Seiberg duality,

Higgsed vacua $\leftrightarrow$ confined vacua

In Higgsed vacua, we have

Wilson	't Hooft	dyonic
perimeter	area	area

while in the confined vacua, we have

Wilson	't Hooft	dyonic
area	**area**	**perimeter**

.

(The red part might sound counter-intuitive,
but is due to a subtle behavior of vacuum branches of $\mathfrak{so}(4)$)

This means that the duality acts as

Wilson	't Hooft	dyonic
$\mathbf{Spin}(2n_c)$	$\mathbf{SO}_+(2n_c)$	$\mathbf{SO}_-(2n_c)$

Wilson	't Hooft	dyonic
$\mathbf{Spin}(2n_c')$	$\mathbf{SO}_+(2n_c')$	$\mathbf{SO}_-(2n_c')$

This was tested by SCI on $(S^3/\mathbb{Z}_k) \times S^1$ [Razamat-Willett 1307.4381]

(It comes with a Mathematica code to generate SCIs. Nice!)

Does the 2-group structure agree?

That is our question.

Higher symmetries in Intriligator-Seiberg duality

Let us remind our discussion of $\mathbf{Spin}(2n_c)$ theory with $2n_f$ flavors.

$$(\text{gauge spinor}) \otimes (\text{gauge spinor}) = \begin{cases} \text{gauge singlet} & (n_c\text{: even}) \\ \text{gauge vector} \xrightarrow{\text{screen}} \textbf{flavor fundamental} & (n_c\text{: odd}) \end{cases}$$

meaning that the $\mathbb{Z}_2$ 1-form symmetry and the $\mathfrak{su}(2n_f)$ 0-form symmetry

$$\begin{cases} \text{remain direct product} & (n_c\text{: even}) \\ \text{form nontrivial 2-group} & (n_c\text{: odd}) \end{cases}$$

How about the $\mathbf{SO}(2n_c)_\pm$ theory with $2n_f$ flavors?

Neglecting the flavor symmetry, the 1-form symmetry is $\mathbb{Z}_2$, because

$$(\text{charge 1 't Hooft line})^{\otimes 2} = \text{charge 2 't Hooft line} \xrightarrow[\text{screening by a dynamical monopole}]{} \text{trivial line}$$

Therefore, the $\mathbb{Z}_2$ 1-form and the flavor 0-form

$$\left\{ \begin{array}{c} \text{remain a direct product} \\ \text{combine into a nontrivial 2-group} \end{array} \right\}$$

depending on **whether the dynamical monopole has**

$$\left\{ \begin{array}{l} \text{charge } +1 \\ \text{charge } -1 \end{array} \right\}$$

under $-1 \in \mathbf{SU}(2n_f)$.

The point is that there can be fermionic zero modes on dynamical monopoles, potentially inducing flavor charges on them.

A famous example is $\mathcal{N}=2$ $\mathfrak{su}(2)$ with n_f flavors.

Each monopole carries fermionic zero modes $\psi_{i=1,\ldots,2n_f}$.

They are quantized into operators satisfying $\{\psi_i, \psi_j\} = \delta_{ij}$, and behave as gamma matrices of $\mathfrak{so}(2n_f)$.

So the monopoles transform in the spinor of $\mathfrak{so}(2n_f)$.

Our situation is similar.

So, our question is now the following:

> *In* $\mathbf{SO(2}\boldsymbol{n_c}\mathbf{)}_\pm$ *with* $\mathbf{2}\boldsymbol{n_f}$ *flavors,*
> *how do dynamical monopoles*
> *transform under* $-\mathbf{1} \in \mathbf{SU(2}\boldsymbol{n_f}\mathbf{)}$*?*

The way to answer it is quite fun in itself,
but I do not go into detail,
as it is something you can do if you live long enough.

We didn't know how to do it in the non-abelian theory itself,
so we choose to break $\mathbf{SO(2}n_c\mathbf{)} \to \mathbf{SO(2)}^{n_c}$
by introducing an adjoint scalar $\mathbf{\Phi}$ and giving it a generic vev.

Then you have 't Hooft-Polyakov monopoles,
zero modes on which can be studied via Callias index theorem.

Done.

The result: the flavor charge of the monopole under $-\mathbf{1} \in \mathbf{SU(2}n_f\mathbf{)}$ is

$$\begin{cases} (-1)^{n_f} & \text{for } \mathbf{SO(2}n_c\mathbf{)}_+ \text{ theory} \\ (-1)^{n_f+n_c} & \text{for } \mathbf{SO(2}n_c\mathbf{)}_- \text{ theory} \end{cases}$$

We conclude the following.
The $\mathbb{Z}_2$ 1-form and the flavor symmetry become:

(n_c, n_f)	**Spin**	$\mathbf{SO}_+$	$\mathbf{SO}_-$
(even, even)	product	product	product
(odd, even)	**2-group**	product	**2-group**
(even, odd)	product	**2-group**	**2-group**
(odd, odd)	**2-group**	**2-group**	product

The I-S duality acts as $\mathfrak{so}(2n_c) \leftrightarrow \mathfrak{so}(2n_f - 2n_c + 4)$,

swaps $\mathbf{Spin} \leftrightarrow \mathbf{SO}_-$, and keeps $\mathbf{SO}_+$.

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(n_c, n_f)	**Spin**	$\mathbf{SO}_+$	$\mathbf{SO}_-$
(even, even)	product	product	product
(odd, even)	**2-group**	product	**2-group**
(even, odd)	product	**2-group**	**2-group**
(odd, odd)	**2-group**	**2-group**	product

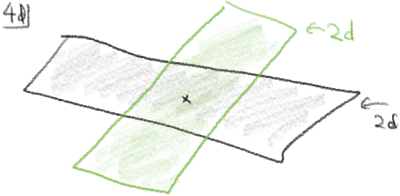
The I-S duality acts as $\mathfrak{so}(2n_c) \leftrightarrow \mathfrak{so}(2n_f - 2n_c + 4)$,

swaps $\mathbf{Spin} \leftrightarrow \mathbf{SO}_-$, and keeps $\mathbf{SO}_+$.

A subtle fermion anomaly

(n_c, n_f)	Spin	SO$_+$	SO$_-$
(even, even)	product	product	product
(odd, even)	2-group	product	2-group
(even, odd)	product	2-group	2-group
(odd, odd)	2-group	2-group	product

'Product' means that the $\mathbb{Z}_2$ 1-form symmetry and the $\mathbb{Z}_2$ 1-form symmetry for $-1 \in \mathbf{SU}(2n_f)$ remains separate.



But there can be a mixed anomaly between these two $\mathbb{Z}_2$ 1-form symmetries.

(n_c, n_f)	**Spin**	$\mathbf{SO_+}$	$\mathbf{SO_-}$
(even, even)	product	product	product
(odd, even)	2-group	product	2-group
(even, odd)	product	2-group	2-group
(odd, odd)	2-group	2-group	product

How do we know which ‘product’ has anomaly and which doesn’t?

(n_c, n_f)	Spin	SO_+	SO_-
(even, even)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
(odd, even)	$\mathbb{Z}_2 \subset \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \subset \mathbb{Z}_4$
(even, odd)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \subset \mathbb{Z}_4$	$\mathbb{Z}_2 \subset \mathbb{Z}_4$
(odd, odd)	$\mathbb{Z}_2 \subset \mathbb{Z}_4$	$\mathbb{Z}_2 \subset \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

How do we know which '$\mathbb{Z}_2 \times \mathbb{Z}_2$' has anomaly and which doesn't?

(n_c, n_f)	**Spin**	$\overset{\text{gauge } \mathbb{Z}_2}{\longrightarrow}$ $\underset{\text{gauge } \mathbb{Z}_2}{\longleftarrow}$	$\mathbf{SO}_+$
(even, even)	$\mathbb{Z}_2 \times \mathbb{Z}_2$		$\mathbb{Z}_2 \times \mathbb{Z}_2$
(odd, even)	$\mathbb{Z}_2 \subset \mathbb{Z}_4$		$\mathbb{Z}_2 \times \mathbb{Z}_2$
(even, odd)	$\mathbb{Z}_2 \times \mathbb{Z}_2$		$\mathbb{Z}_2 \subset \mathbb{Z}_4$
(odd, odd)	$\mathbb{Z}_2 \subset \mathbb{Z}_4$		$\mathbb{Z}_2 \subset \mathbb{Z}_4$

How do we know which '$\mathbb{Z}_2 \times \mathbb{Z}_2$' has anomaly and which doesn't?

But as I reviewed, it is a general fact that

$$\begin{tikzcd}
\begin{array}{c}\mathbb{Z}_2 \times \mathbb{Z}_2 \\ \text{without anomaly}\end{array} \arrow[r, bend left, "\text{gauge } \mathbb{Z}_2"] & \begin{array}{c}\mathbb{Z}_2 \times \mathbb{Z}_2 \\ \text{without anomaly}\end{array} \\
\begin{array}{c}\mathbb{Z}_2 \subset \mathbb{Z}_4 \\ \text{without anomaly}\end{array} & \begin{array}{c}\mathbb{Z}_2 \times \mathbb{Z}_2 \\ \text{with anomaly}\end{array} \arrow[l, bend left, "\text{gauge } \mathbb{Z}_2"]
\end{tikzcd}$$

Therefore, it must be that

gauge $\mathbb{Z}_2$

(n_c, n_f)	**Spin**	$\mathbf{SO_+}$
(even, even)	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anom.	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anom.
(odd, even)	$\mathbb{Z}_2 \subset \mathbb{Z}_4$ without anom.	$\mathbb{Z}_2 \times \mathbb{Z}_2$ with anom.
(even, odd)	$\mathbb{Z}_2 \times \mathbb{Z}_2$ with anom.	$\mathbb{Z}_2 \subset \mathbb{Z}_4$ without anom.
(odd, odd)	$\mathbb{Z}_2 \subset \mathbb{Z}_4$ with anom.	$\mathbb{Z}_2 \subset \mathbb{Z}_4$ with anom.

gauge $\mathbb{Z}_2$

Therefore, it must be that

$$\mathbf{Spin} \underset{\text{gauge } \mathbb{Z}_2}{\overset{\text{gauge } \mathbb{Z}_2}{\rightleftarrows}} \mathbf{SO}_+$$

(n_c, n_f)	**Spin**	$\mathbf{SO}_+$
(even, even)	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anom.	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anom.
(odd, even)	$\mathbb{Z}_2 \subset \mathbb{Z}_4$ without anom.	$\mathbb{Z}_2 \times \mathbb{Z}_2$ with anom.
(even, odd)	$\mathbb{Z}_2 \times \mathbb{Z}_2$ with anom.	$\mathbb{Z}_2 \subset \mathbb{Z}_4$ without anom.
(odd, odd)	$\mathbb{Z}_2 \subset \mathbb{Z}_4$ with anom.	$\mathbb{Z}_2 \subset \mathbb{Z}_4$ with anom.

(n_c, n_f)	Spin
(even, even)	$\mathbb{Z}_2 \times \mathbb{Z}_2$ without anom.
(even, odd)	$\mathbb{Z}_2 \times \mathbb{Z}_2$ with anom.

The background for $\mathbb{Z}_2$ **1-form** is w_2
controlling the lift from $\mathbf{SO}(2n_c)$ to $\mathbf{Spin}(2n_c)$,

The background for '**flavor $\mathbb{Z}_2$ 1-form**' is a_2
controlling the lift from $\mathbf{SU}(2n_f)/\mathbb{Z}_2$ to $\mathbf{SU}(2n_f)$.

(n_c, n_f)	fermion in $\mathbf{2n_c} \otimes \mathbf{2n_f}$
(even, even)	without anom.
(even, odd)	with anom.

This means that the bifundamental fermion
in $\mathbf{2n_c} \otimes \mathbf{2n_f}$ of $[\mathbf{SO(2n_c)} \times \mathbf{USp(2n_f)}]/\mathbb{Z}_2$
should have the anomaly

$$\int_{5\mathrm{d}} a_2 \beta w_2$$

when n_c is odd.

This is a rather subtle global anomaly!

In the past ten years, the **theory of global anomalies** was perfected using **spin bordism groups**.

In principle, given a fermion transforming in a representation of a group, we should now be able to compute its anomaly using the η invariant.

Then, *in principle*, higher symmetry structures of the theories in question should directly follow.

But it is exactly a thing which is 言之易而行之難 (easier said than done).

The spin bordism group of $[\mathbf{SO(2}n_c\mathbf{)} \times \mathbf{USp(2}n_f\mathbf{)}]/\mathbb{Z}_2$
is hard to compute.

Even the cohomology of $[\mathbf{SO(2}n_c\mathbf{)} \times \mathbf{USp(2}n_f\mathbf{)}]/\mathbb{Z}_2$
is hard to obtain.

Even supposing that the bordism group is known,
finding concrete manifolds with bundles representing them
is extremely hard.

Evaluating the $\boldsymbol{\eta}$ invariants is hard.

We struggled with these issues for more than a year.

Various partial results slowly suggested us the big picture.

Only in the last few months,
we came up with the indirect method I described today.

Anyway...

So the big picture requires that

$(\boldsymbol{n_c}, \boldsymbol{n_f})$	fermion in $\mathbf{2}\boldsymbol{n_c} \otimes \mathbf{2}\boldsymbol{n_f}$
(even, even)	without anom.
(even, odd)	with anom.

where the anomaly is

$$\int_{5\mathrm{d}} \boldsymbol{a_2 \beta w_2}.$$

We were able to confirm this for $[\mathbf{SO(4)} \times \mathbf{SU(2)}]/\mathbb{Z}_\mathbf{2}$.

For more general cases, we could only perform checks.

If you're interested, please have a look at the paper.

Summary

We reviewed 1-form symmetries and 2-groups.

We studied them in the case of $\mathfrak{so}(2n_c)$ with $2n_f$ flavors.

They are mapped as expected under the Intriligator-Seiberg duality.

Our results indicate that there are subtle global anomalies of fermions.

Any questions?