Recent developments on 4d $\mathcal{N}=3$ theories

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- $\mathcal{N}=4$ theories are very important.
  S-duality, Prototypical AdS/CFT, integrability, …

- $\mathcal{N}=2$ theories are also very important.
  Seiberg-Witten theory, class S theory, wall-crossing, …

- We don't hear $\mathcal{N}=3$ discussed very often.

- In the past, it was often said that $\mathcal{N}=3 \Rightarrow \mathcal{N}=4$, but that was for **Lagrangian field theories**.

- In the last several years, we learned that **non**-Lagrangian theories are everywhere.

- Last December, [Aharony-Evtikhiev 1512.03524] studied 4d $\mathcal{N}=3$ superconformal multiplets and studied what $\mathcal{N}=3$ theories look like, **if they exist.**

- Two weeks later,
  in [Gacía-Etxebarria and Regalado, 1512.06434],
  genuine $\mathcal{N}=\mathbf{3}$ theories were constructed **using F-theory**.

- I was very excited and I wrote two follow-ups,
  [1602.01503 with Nishinaka] and
  [1602.08638 with Aharony].
- I'd like to report on these subjects.

- also $\exists$ two related works
  [Argyres-Lotito-Lu-Martone, 1601.00011, 1602.02764]
  [Imamura-Yokoyama, 1603.00851].

Review of the old argument

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F-theory construction, part I

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QFT analysis

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F-theory construction, part II

- Analysis of susy multiplets can be found in [Wess-Bagger].
- (spin $\leq \mathbf{1}$ & CPT-inv & $\mathcal{N}=\mathbf{3}$) implies $\mathcal{N}=\mathbf{4}$.
- They only analyzed the multiplets, not interactions.
- At this point $\exists$ possibility of genuine $\mathcal{N}=\mathbf{3}$ interactions.

- Assume $\mathcal{N}=\mathbf{2}$.
  All possible Lagrangians up to lowest derivatives easily written.
  One goes over the list, and finds $\mathcal{N} > \mathbf{2}$ implies $\mathcal{N}=\mathbf{4}$.

- Still $\exists$ possibility of genuine $\mathcal{N}=\mathbf{3}$ higher-derivative interactions.
- $\exists$ genuine $\mathcal{N}=\mathbf{3}$ supergravity with two derivatives.

- Anyway, this says nothing about non-Lagrangian theories!

- By the way, in 3d,
- [Bashkirov 1108.4081] analyzed the **superconformal** multiplets and found $\mathcal{N}=7$ implies $\mathcal{N}=8$.
- So, no similar question in 3d.

- [Aharony-Evtikhiev 1512.03524] studied 4d $\mathcal{N}=3$ **superconformal** multiplets.
- They found, for example, an $\mathcal{N}=3$ preserving **marginal deformation** is in the same multiplet with another supercharge.
  $\to$ **the theory enhances** to $\mathcal{N}=4$.
- Put it differently, genuine $\mathcal{N}=3$ theories are intrinsically **isolated**.
- Field-theoretical construction sounds very difficult.
- But there's an easy stringy construction!

Review of the old argument

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F-theory construction, part I

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QFT analysis

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F-theory construction, part II

- Let's construct 4d $\mathcal{N}=\mathbf{3}$ theory.
- We need to keep $\mathbf{3/4}$ of SUSY.
- Do you know any other system that preserves $\mathbf{3/4}$ of SUSY?

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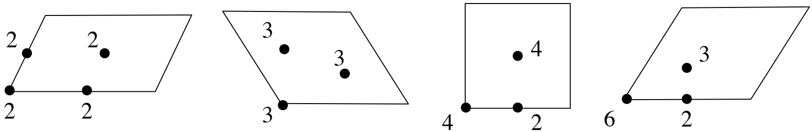
- We need to keep $3/4$ of SUSY.
- Do you know any other system that preserves $3/4$ of SUSY?

- **The ABJM theory** !
- M-theory on $\mathbb{C}^4/\mathbb{Z}_k$, probed by M2s.
- 3d $\mathcal{N}=8$ when $k=1,2$, 3d $\mathcal{N}=6$ when $k \geq 3$.

- We can imitate the construction in 4d.
- Just need to consider F-theory on $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_k$, probed by D3s.

- T^2 can have $\mathbb{Z}_k$ action only for $k = 1, 2, 3, 4, 6$.
- For $k = 1, 2$,
  T^2 can be arbitrary.
- For $k = 3, 4, 6$,
  T^2 needs to have a particular shape:



$k = 2$ $k = 3$ $k = 4$ $k = 6$

where the numbers show the orders of the fixed points.

- Consider F-theory on $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_k$, probed by N D3s.
- For $k = 1$, this gives 4d $\mathcal{N}{=}4$ $\mathrm{U}(N)$ SYM.
- The shape of T^2 is arbitrary, and gives τ.

- Consider F-theory on $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_k$, probed by N D3s.
- For $k = 2$, this $\mathbb{Z}_2$ quotient is the standard orientifold 3-planes.
- Has variants:

$$\mathrm{O3}^-, \quad \widetilde{\mathrm{O3}}^-, \quad \mathrm{O3}^+, \quad \widetilde{\mathrm{O3}}^+.$$

- They give rise to 4d $\mathcal{N}=4$ SYM with gauge groups

$$\mathrm{SO}(2N), \quad \mathrm{SO}(2N+1), \quad \mathrm{USp}(2N), \quad \mathrm{USp}(2N),$$

  respectively.
- The shape of T^2 is arbitrary, and gives τ.

- Consider F-theory on $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_k$.
- For $k = 3, 4, 6$, we have a **something-fold** that generalizes **orientifolds**.

- Will have **variants**.

- Probing this setup with N D3-branes, we have genuine $\mathcal{N}{=}3$ theory, labeled by k, N, and **another label for variants**.

- The shape of T^2 is **fixed, with no marginal parameter**.

- This is basically the content of [García-Etxebarria and Regalado].

- We would like to understand the variants,
  and to get some idea of the field theoretical properties.

Review of the old argument

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F-theory construction, part II

- Let's first see how much we can say just from field theory.
- For simplicity, consider the rank-1 $\mathcal{N}=3$ theories.
- Here the **rank** $=$ dim. of Coulomb branch as $\mathcal{N}=2$ theories.

- rank-1 $\mathcal{N}=2$ SCFT $\Rightarrow$

  its Seiberg-Witten geometry is a T^2 fibration over $\mathbb{C}$ with τ constant up to monodromy $\Rightarrow$

  Kodaira classification says it's either

$$II, \quad III, \quad IV, \quad I_0^*, \quad IV^*, \quad III^*, \quad II^*.$$

- Correspondingly, the **Coulomb branch** operator u has dimension

$$6/5, \quad 4/3, \quad 3/2, \quad 2, \quad 3, \quad 4, \quad 6.$$

- Next, regard the rank-1 $\mathcal{N}=3$ theory as an $\mathcal{N}=2$ theory and consider its **Higgs branch**.

- 1-dim. hyperkähler cone $\Rightarrow$ ALE space $\mathbb{C}^2/\Gamma$.

- $\mathcal{N}=3$ superconformal mult. contains
  $\mathcal{N}=2$ superconformal mult. together with
  $\mathcal{N}=2$ $\mathrm{U}(1)$ flavor current multiplet.

- This $\mathrm{U}(1)$ needs to act on $\mathbb{C}^2/\Gamma \Rightarrow \Gamma = \mathbb{Z}_\ell$.

- Chiral operators on $\mathbb{C}^2/\mathbb{Z}_\ell$ generated by

$$zw = t^\ell$$

  with z, w of dim. ℓ and t of dim. 2.

- Higgs branch operator z: dimension $\ell \in \mathbb{Z}$
- Coulomb branch operator u: dimension is one of

$$6/5, \quad 4/3, \quad 3/2, \quad 2, \quad 3, \quad 4, \quad 6.$$

- $\mathcal{N}=3$ SUSY relates the two $\Rightarrow \ell = 2, 3, 4, 6$.
- $\mathcal{N}=3$ SUSY says that if $\ell = 2$ it enhances to $\mathcal{N}=4$.

- Genuine $\mathcal{N}=3 \Rightarrow \ell = 3, 4, 6$.

- Coulomb branch $= \mathbb{C}^1/\mathbb{Z}_\ell$, Higgs branch $= \mathbb{C}^2/\mathbb{Z}_\ell$
- Combines to the $\mathcal{N}{=}3$ moduli space

$$\mathbb{C}^3/\mathbb{Z}_\ell,$$

for $\ell = 3, 4, 6$.

- Agrees with what F-theory tells us.

- if we go to a generic point on $\mathbb{C}^3/\mathbb{Z}_\ell$, it's just a theory of $\mathcal{N}{=}4$ $\mathrm{U}(1)$ vector multiplet.
- $\mathbb{Z}_\ell$ monodromy reduces the susy to $\mathcal{N}{=}3$.

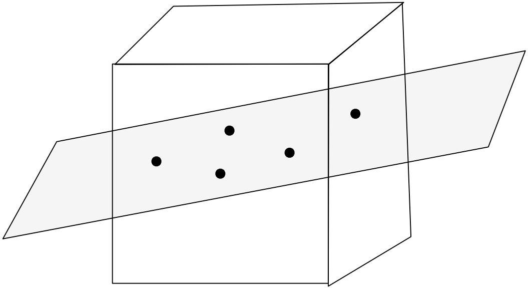
- What are their a and c?
- [Aharony-Evtikhiev 1512.03524]: $\mathcal{N}=3$ implies $a = c$.
- [Shapere-YT 0804.1957]:

$$2a - c = \sum_u \frac{2\Delta(u) - 1}{4}.$$

- Plugging in $\Delta(u) = \ell = 3, 4, 6$, we have

$$a = c = \frac{2\ell - 1}{4}.$$

- We can also study the 2d chiral algebra associated to these $\mathcal{N}=3$ theory in the sense of [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees, 1312.5344]
- Take any $\mathcal{N}=2$ SCFT. Consider correlation functions such that all operators are on a single $\mathbb{R}^2$:



- After a small twist, they satisfy the axioms of 2d chiral algebra, with $c_{2\mathrm{d}} = -12c_{4\mathrm{d}}$.

- When we start from 4d $\mathcal{N}=3$, the twist breaks some susy, resulting in 2d $\mathcal{N}=2$.
- We had Higgs branch operators $zw = t^\ell$.

- 4d $\mathcal{N}=2$ superconformal multiplet together with Higgs op. t
  $\Rightarrow$ 2d $\mathcal{N}=2$ **super Virasoro** T with

$$c_{\text{2d}} = -12c_{\text{4d}} = -3(2\ell - 1)$$

- Higgs op. z and w
  $\Rightarrow$ 2d **chiral primary** Z and **anti-chiral primary** W,
  both with dimension $h = \ell/2$.
- Are there such super W-algebras?

- **Yes**.

- To see this, use $\mathcal{N}=2$ version of `OPEdefs` of [Krivonos-Thielemans hep-th/9512029] to check the Jacobi identity.

- One finds, for each $\ell = 3, 4, 5, \ldots$, only for

$$c_{\text{2d}} = -3(2\ell - 1)$$

  $\exists$ super W algebra with the required properties.

- The Jacobi only **closes thanks to special null states** at this particular value of the central charge.
- Other null states automatically implies $ZW \sim T^{\ell}$.

- ℓ is not restricted to $3, 4, 6$.
  Not contradictory, but not very satisfying ...

Review of the old argument

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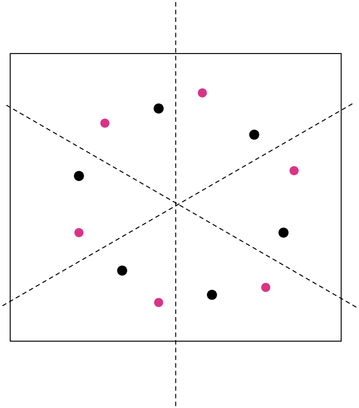
F-theory construction, part II

- How about the higher-rank cases?
  How to see the variants of **something-folds** for $\boldsymbol{k = 3, 4, 6}$,
  analogous to the distinction of $\mathrm{SO}(\boldsymbol{2N})$ and $\mathrm{USp}(\boldsymbol{2N})$ for $\boldsymbol{k = 2}$?

- To study this issue, consider D3-branes on $\mathbb{C}/\mathbb{Z}_{\boldsymbol{k}} \subset \mathbb{C}^{\boldsymbol{3}}/\mathbb{Z}_{\boldsymbol{k}}$.

- This corresponds to restricting attention
  to the $\boldsymbol{\mathcal{N}{=}2}$ Coulomb branch.

- Say we have N points on $\mathbb{C}/\mathbb{Z}_k \subset \mathbb{C}^3/\mathbb{Z}_k$:



- Coordinates $z_1, \ldots, z_N \in \mathbb{C}$, with kN images.
  Coordinates of the "Cartan" of the "gauge group".

- Invariance under

$$z_i \leftrightarrow z_j$$
$$z_i \mapsto \gamma z_i \quad (\gamma^k = 1),$$

  the "Weyl group".
- "Weyl-invariant" combinations are sym. poly. of ${z_i}^k$, with dimension

$$k, \quad 2k, \quad 3k, \quad \cdots \quad , Nk.$$

- Taking $k = 2$, this gives the gauge invariants of $\mathrm{SO}(2N+1)$ or $\mathrm{USp}(2N)$, but **not** those of $\mathrm{SO}(2N)$.

- Weyl group of $\mathrm{SO}(2N+1)$ is

$$z_i \leftrightarrow z_j$$
$$z_i \mapsto -z_i \quad \text{fixing others}$$

- Weyl group of $\mathrm{SO}(2N)$ is

$$z_i \leftrightarrow z_j$$
$$(z_i, z_j) \mapsto (-z_i, -z_j) \quad \text{fixing others}$$

  a subgroup of the above group.

- The top-dimension invariant of the former, the **determinant**

$$(z_1 \ldots z_N)^2$$

  is the square of the top-dimension invariant of the latter, the **Pfaffian**

$$(z_1 \ldots z_N).$$

- Similarly, the group $G(N, 1, k)$

$$\begin{aligned} z_i &\leftrightarrow z_j \\ z_i &\mapsto \gamma z_i \quad (\gamma^k = 1), \end{aligned}$$

has subgroups $G(N, p, k)$ for $k = p\ell$:

$$\begin{aligned} z_i &\leftrightarrow z_j \\ z_i &\mapsto \gamma^p z_i && \text{fixing others} \\ (z_i, z_j) &\mapsto (\gamma z_i, \gamma^{-1} z_j) && \text{fixing others} \end{aligned}$$

- The top-dimension invariant of the former, the "**determinant**"

$$(z_1 \dots z_N)^k$$

is the p-th power of
the top-dimension invariant of the latter, the "**Pfaffian**"

$$(z_1 \dots z_N)^\ell$$

- In general, the Coulomb branch operators of $\mathcal{N}=2$ SCFTs are believed to have **no relation and form a polynomial ring**.
- Suppose a finite group G acts on a vector space V.
  When the invariants form a **polynomial ring**?
  Answer: when G is a **complex reflection group**.

- $G(N,p,k)$ is one of those. Dimensions of the invariants:

$$k, \quad 2k, \quad \cdots \quad ,(N-1)k; \quad \ell k$$

  where the last invariant is $(z_1 \dots z_N)^\ell$, where $\ell p = k$.

- So the **variants of those something-fold would be labeled** by $\ell | k$.
- The question is which ℓ is realized in F-theory.

$\boldsymbol{k = 2}$	$\boldsymbol{\ell = 1}$	$\boldsymbol{\ell = 2}$				
$\boldsymbol{k = 3}$	$\boldsymbol{\ell = 1}$		$\boldsymbol{\ell = 3}$			
$\boldsymbol{k = 4}$	$\boldsymbol{\ell = 1}$	$\boldsymbol{\ell = 2}$		$\boldsymbol{\ell = 4}$		
$\boldsymbol{k = 6}$	$\boldsymbol{\ell = 1}$	$\boldsymbol{\ell = 2}$	$\boldsymbol{\ell = 3}$		$\boldsymbol{\ell = 6}$	

- Those marked in **green** do not exist. How do we see that?

- Variants of something-folds are classified by G_3 and H_3 fluxes.
- For orientifolds, it was worked out in [Witten, hep-th/9805112].
- Consider $S^5/\mathbb{Z}_2$ surrounding the orientifold. We need to consider

$$H^3(S^5/\mathbb{Z}_2, \tilde{\mathbb{Z}} \oplus \tilde{\mathbb{Z}})$$

  where the tilde over $\mathbb{Z}$ means that
  the $\mathbb{Z}_2$ action on S^5 flips the sign of G_3 and H_3.

- This cohomology group turns out to be $\mathbb{Z}_2 \oplus \mathbb{Z}_2$,
  corresponding to $\mathrm{O3}^-$, $\widetilde{\mathrm{O3}}^-$, $\mathrm{O3}^+$, $\widetilde{\mathrm{O3}}^+$.

- Under $SL(2,\mathbb{Z})$, $\mathrm{O3}^-$ is a single orbit
  and **the other three** form another orbit. Thus two orbits.
- **Two variants** at the level of the spectrum of gauge invariants.

- For $k = 3, 4, 6$, we need to consider

$$H^3(S^5/\mathbb{Z}_k, (\mathbb{Z} \oplus \mathbb{Z})_\rho)$$

where $\rho : \mathbb{Z}_k \to SL(2, \mathbb{Z})$ specifies how $\mathbb{Z}_k$ acts on G_3 and H_3.
- They turn out to be $\mathbb{Z}_3$, $\mathbb{Z}_2$, $\mathbb{Z}_1$ respectively.

- For $k = 3$,

$$-1, 0, +1 \in \mathbb{Z}_3$$

but ± 1 are equivalent under $SL(2, \mathbb{Z})$, thus **two variants**.
- For $k = 4$, **two variants**.
- For $k = 6$, **one variant** only.

- For $k = 3$, **two variants**.
- For $k = 4$, **two variants**.
- For $k = 6$, **one variant** only.

$k = 2$	$\ell = 1$	$\ell = 2$			
$k = 3$	$\ell = 1$		$\ell = 3$		
$k = 4$	$\ell = 1$	$\ell = 2$		$\ell = 4$	
$k = 6$	$\ell = 1$	$\ell = 2$	$\ell = 3$		$\ell = 6$

- How do I know that those marked in **blue** are the ones given in F-theory?

- One way to see this is as follows.
- Consider the holographic dual $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_k$.
- The operator $(z_1 \cdots z_N)^\ell$ corresponds to a D3-brane wrapped around $S^3/\mathbb{Z}_k \subset S^5/\mathbb{Z}_k$ ℓ times.
- Nontrivial flux $\in H^3(S^5/\mathbb{Z}_k, (\mathbb{Z} \oplus \mathbb{Z})_\rho)$ obstructs this.
- For $k = 2$, studied in [Witten, hep-th/9805112].

- For $k = 6$, $H^3 = \mathbb{Z}_1$, so there's no nontrivial flux to start with. So $\ell = 1$ always allowed.
- For $k = 3$, $H^3 = \mathbb{Z}_3$. When the flux is zero, $\ell = 1$. When the flux is nonzero, only $\ell = 3$ allowed.
- For $k = 4$, $H^3 = \mathbb{Z}_2$. When the flux is zero, $\ell = 1$. When the flux is nonzero, only $\ell = 4$ allowed, after a careful consideration.

- Another way to see this is as follows.
- Consider the holographic dual AdS$_5 \times S^5/\mathbb{Z}_k$ again.
- It's easily seen that

$$a \sim c \sim \frac{k(N+\epsilon)^2}{4} + O(N^0)$$

where ϵ is the intrinsic D3-charge of the something-fold.

- Meanwhile, we know

$$a = c, \qquad 2a - c = \sum_u \frac{2\Delta(u) - 1}{4}$$

with $\Delta(u)$'s given by

$$k, \quad 2k, \quad \cdots \quad , (N-1)k; \quad \ell k$$

- Comparing with

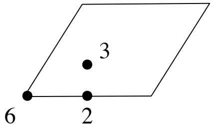
$$a \sim c \sim \frac{k(N+\epsilon)^2}{4} + O(N^0)$$

one finds

$$\epsilon = \frac{2\ell - k - 1}{12}.$$

- This intrinsic D3-charge of the something-fold can be computed independently, by compactifying the system on S^1 and use M-theory.

- Take e.g. $k = 6$. We have $\mathbb{R}^3 \times (\mathbb{C}^3 \times T^2)/\mathbb{Z}_6$.
- The Z_6 action on T^2 is given by



- So there are $\mathbb{C}^4/\mathbb{Z}_2$, $\mathbb{C}^4/\mathbb{Z}_3$, $\mathbb{C}^4/\mathbb{Z}_6$ singularities.

- A $\mathbb{C}^4/\mathbb{Z}_s$ singularity is known to have M2-charge

$$-\frac{1}{24}(s-\frac{1}{s}).$$

- We have

$$-\frac{1}{24}\left[(2-\frac{1}{2})+(3-\frac{1}{3})+(6-\frac{1}{6})\right]=-\frac{5}{12}.$$

- This agrees with

$$\epsilon=\frac{2\ell-k-1}{12}$$

with $\ell=1$, $k=6$.

- All other examples work out nicely.

Conclusions

- Genuine $\mathcal{N}=3$ theories exist!
- Some properties were understood,
  a and c, 2d chiral algebra, variants.
- Many properties still to be determined,
  e.g. Schur indices and more general superconformal indices.

- The quotients of [Gacía-Etxebarria and Regalado] can be generalized to $\mathcal{N}=2$.
  $\Rightarrow$ a lot of novel $\mathcal{N}=2$ SCFTs so far unknown.

- Many things to do!