Physics and Algebraic Topology

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Mathematics is unreasonably effective for us physicists,
as Wigner famously mentioned. excerpt

But the usefulness depends on the subfields of math.

Ordinary/partial differential equations are obviously effective.

Group theory is also obviously effective to describe **symmetry**.

Differentiable manifolds are the basis of **general relativity**.

Algebraic geometry?	Only in string theory.
Number theory?	Not much.
Mathematical logic?	Hmm... but see a recent paper here

How about algebraic topology?

Some use have been made in the past.

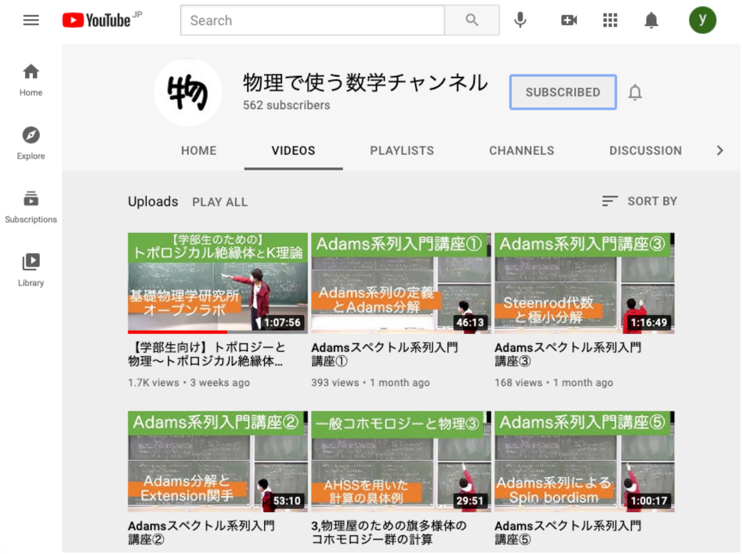
Notably, **homotopy groups** were used
to understand **topological solitons in 1970s.**

Not much else has been used until late 1990s,
when string theorists started to use K-theory.

(We can debate whether string theory is physics, though.)

More recently, in the last 10 years,
physicists started to use algebraic topology **more fully**.

A youtube channel run by a grad student in Kyoto:



https://www.youtube.com/channel/UCi4ZotOnAla-loruLQkeyMw/videos

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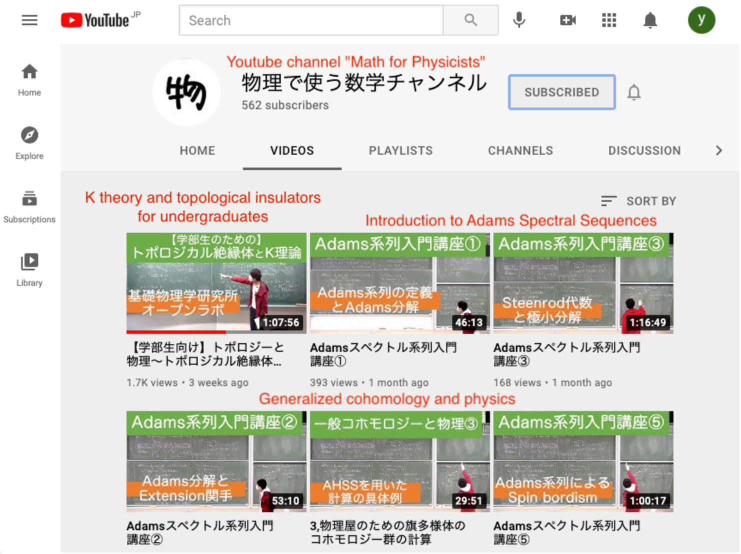

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The poster of a physics student group in 五月祭 (the annual campus festival in Hongo):

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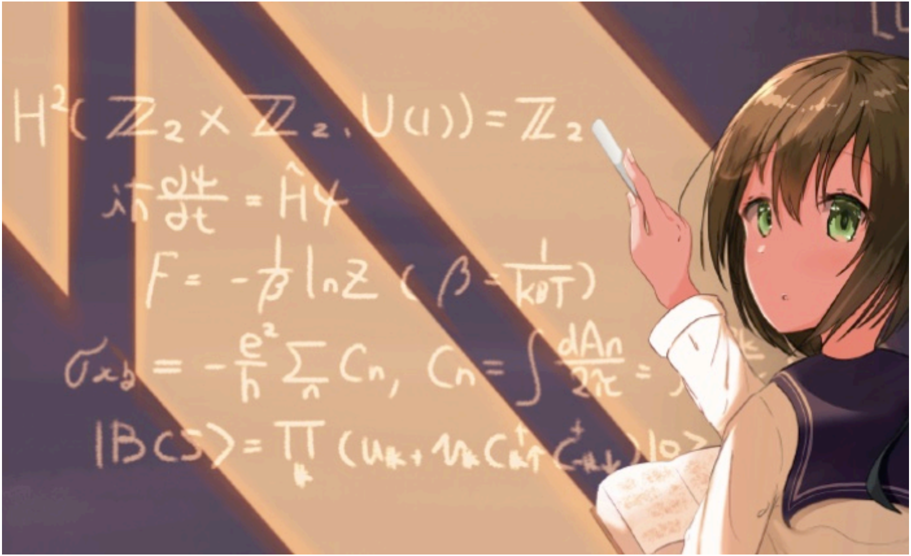

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Today I would like to review the relationship
between physics and algebraic topology.

Concrete homotopy groups are useful in studying topological solitons.

(math: 1930s, physics: 1970s)

Anderson duals of bordism homologies classify SPT phases.

(math: 1960s, physics: 2010s)

TMF and 2d supersymmetric field theories

(math: 2000s, physics: 2020s)

We're trailing behind, but slowly catching up.

Pre-history

up to 1970s

Math side

Hopf invariant / fibration (1931)

$$S^3 = \{(a,b) \in \mathbb{C}^2 \mid |a|^2 + |b|^2 = 1\} \\ \to S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

where

$$(a,b) \mapsto (2\,\mathrm{Re}\, a\bar{b}, 2\,\mathrm{Im}\, a\bar{b}, |a|^2 - |b|^2)$$

(a,b) and $e^{i\theta}(a,b)$ map to the same point on S^2.

S^3 is an S^1 bundle over S^2 with $\int_{S^2} c_1 = 1$.

(If you download the slides, texts in purple are linked to journal webpages etc.)

Physics side

Dirac's quantization condition (1931)

The magnetic charge of a magnetic monopole is an integer multiple of a fixed constant.

Modern paraphrase of Dirac's argument:

Wavefunction of an electron is a section of a complex line bundle $\mathcal{L}$ *over space.*

Electromagnetic field is the $U(1)$ *connection of this line bundle, and the magnetic field strength* F *is its curvature . Therefore,*

$$\int_{S^2} \frac{F}{2\pi} = \int_{S^2} c_1(\mathcal{L}) \in \mathbb{Z}.$$

Math side

Steady progress in algebraic topology.

Stiefel-Whitney / Pontryagin / Chern classes	('30s – '40s)
Eilenberg-Steenrod axiom for (co)homology	(1945)
$H^*(G) := H^*(BG)$ for finite G	(Eilenberg-Mac Lane 1947)
Bordism groups	(Pontryagin, Thom '50s)
Adams spectral sequence	(1958)
K-theory	(Atiyah-Hirzebruch 1959, 1961)

Physics side

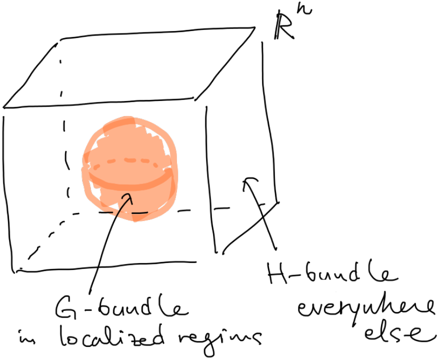
Not much happens in this area until 1970s,

when some concrete homotopy groups were used

to study topological solitons.

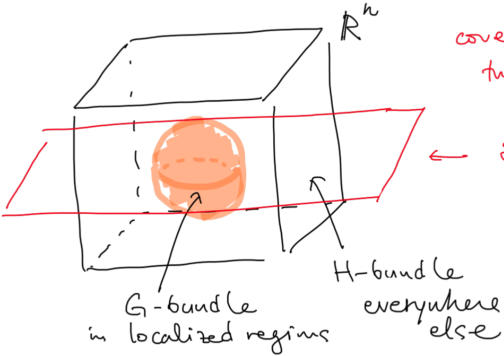
What are topological solitons?

A G-symmetric system can come with a G-bundle.

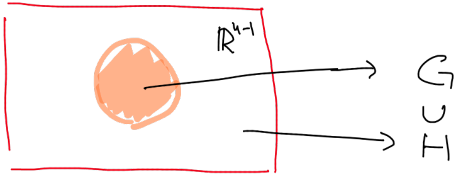


There are situations where having an H-bundle for $H \subset G$ is energetically more favorable.

G is said to be "spontaneously broken to H" in physics.



cover by two patches and glue in the middle.



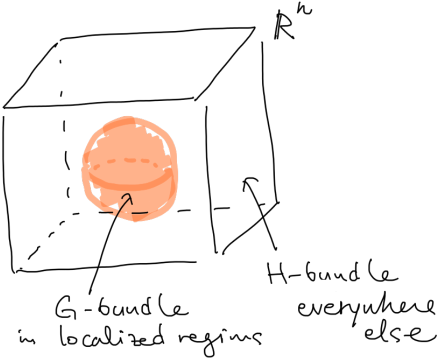
i.e.

$$\begin{array}{ccc} D^{n-1} & \to & G \\ \cup & & \cup \\ S^{n-2} & \to & H \end{array}$$

which determines a class in

$$\pi_{n-1}(G/H).$$

A topological soliton

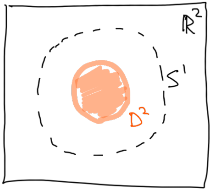


gives a class in

$$\pi_{n-1}(G/H).$$

Example 1

In a superconducting material,
the electromagnetic $G = U(1)$ symmetry is broken to $H = \{\pm 1\}$.



$U(1)$-bundle in the interior; $\{\pm 1\}$-bundle outside.

Measured by $n \in \pi_1(U(1)/\{\pm 1\}) = \mathbb{Z}$, which translates to the magnetic flux

$$\int_{D^2} \frac{F}{2\pi} = \int_{D^2} c_1 = \frac{n}{2}.$$

Known as Abrikosov vortex (1957) in condensed matter physics and Nielsen-Olsen vortex (1973) in high energy physics.

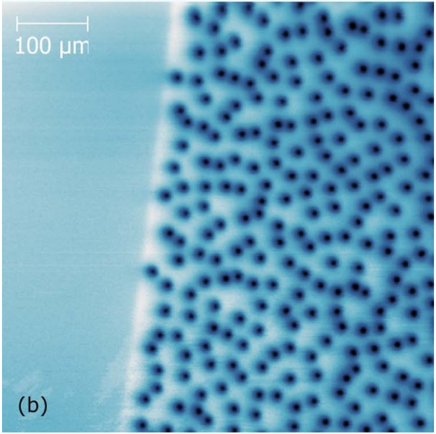
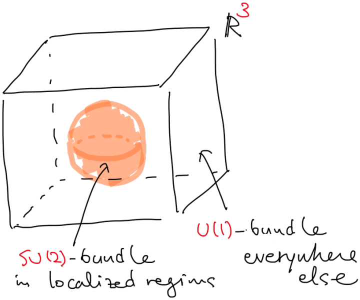


Figure 1 | Images of vortices in 200 nm thick YBCO film taken by Scanning SQUID Microscopy after field cooling at 6.93 μT to 4 K. (b) is taken after heating above T_c and re-cooling. The sample edge at the left side of the images is used as a reference for scan location.

Wells, Pan, Wang, Fedoseev, Hilgenkamp (2015)

Example 2

Taking $G = SU(2)$ and $H = U(1) \subset SU(2)$, you can consider



which is classified by

$$\pi_2(SU(2)/U(1)) = \pi_2(S^2) = \mathbb{Z}.$$

Known as the 't Hooft-Polyakov monopole (1974).

Example 3

The A-phase of the superfluid helium-3 (Osheroff-Richardson-Lee 1972) is characterized by

$$G = SO(3) \times SO(3) \times U(1) \curvearrowright \mathbb{C}^3 \otimes \mathbb{C}^3$$

and

$$H = \text{stabilizer at } \mathbf{e_1} \otimes (\mathbf{e_2} + i\mathbf{e_3})$$

so we have

$$\begin{aligned} \text{vortices}: \quad & \pi_1(G/H) = \mathbb{Z}/4\mathbb{Z}, \\ \text{“monopoles”}: \quad & \pi_2(G/H) = \mathbb{Z}. \end{aligned}$$

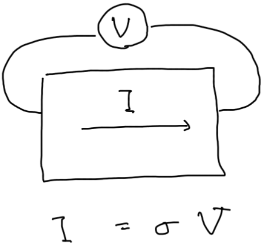
Furthermore, $\pi_1(G/H)$ acts nontrivially on $\pi_2(G/H)$.

Volovik-Mineev (1976)

Middle ages

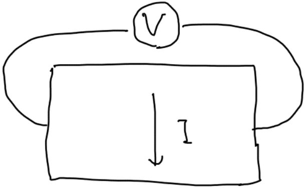
1980s–2000s

What you learn in high school:



σ is called the conductivity.

In a two-dimensional material, this can also happen:



$I = \sigma_H V$

σ_H is called the Hall conductivity.

Surprising discovery of von Klitzing, Dorda, Pepper (1980):

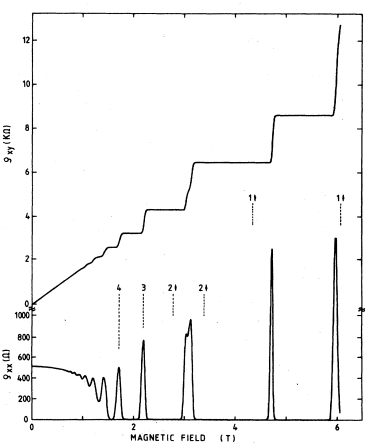


FIG. 14. Experimental curves for the Hall resistance $R_H = \rho_{xy}$ and the resistivity $\rho_{xx} \sim R_x$ of a heterostructure as a function of the magnetic field at a fixed carrier density corresponding to a gate voltage $V_g = 0$ V. The temperature is about 8 mK.

Figure is taken from a slightly later review, von Klitzing (1986)

When the ordinary conductivity σ vanishes, i.e. the system is **gapped**, the Hall conductivity has the universal value

$$\sigma_H = \nu \frac{e^2}{h}, \qquad \nu \in \mathbb{Z}$$

where e is the electric charge of the electron
and h is the Planck constant.

Called the **integer quantum Hall effect**.

This is now the accepted method to calibrate
the experimental apparatus against the declared value of e^2/h.

Why is ν an integer?

There are both **microscopic** understanding
and **macroscopic** understanding.

Microscopic understanding is briefly given in the appendix jump.

Let us concentrate on the **macroscopic** understanding.

Consider an idealized situation where the quantum Hall material **fills the entire $2+1$ dimensional spacetime M.**

M comes with a $U(1)$ bundle $\mathcal{L}$ with connection A describing the electromagnetic field.

The integer quantum Hall material is gapped with unique ground state.

This means that the **system determines the partition function**

$$Z(M, A) \in U(1).$$

When the $U(1)$ bundle is topologically trivial, A is a one-form. The standard Kubo formula says that the coefficient ν in

$$\sigma_H = \nu \frac{e^2}{h}$$

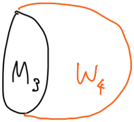
appears in the partition function as

$$Z(M, A) = \exp(i \frac{\nu}{4\pi} \int_M AdA).$$

How do we know that ν is an integer?

We use the fact that AdA is not well-defined for a topologically non-trivial $U(1)$-bundle $\mathcal{L}$.

Given



we have

$$i\frac{\nu}{4\pi}\int_{M_3} AdA = i\frac{\nu}{4\pi}\int_{W_4} FF = \pi i\nu \int_{W_4} c_1(\mathcal{L})^2.$$

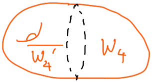
(Note $F = dA$ and $c_1 = F/(2\pi)$.)

The RHS makes sense for topologically nontrivial $\mathcal{L}$,
but looks like it depends on W_4.

Let us compare the two different choices $\boldsymbol{W_4}$ and $\boldsymbol{W_4'}$:

The difference is

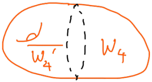
$$\frac{\exp(\pi i\nu \int_{W_4} c_1(\mathcal{L})^2)}{\exp(\pi i\nu \int_{W_4'} c_1(\mathcal{L})^2)} = \exp\Big(\pi i\nu \int$$



$$c_1(\mathcal{L})^2\Big)$$

So we need to ask:

$$\exp\Big(\pi i \nu \int$$



$$c_1(\mathcal{L})^2\Big) \stackrel{?}{=} 1$$

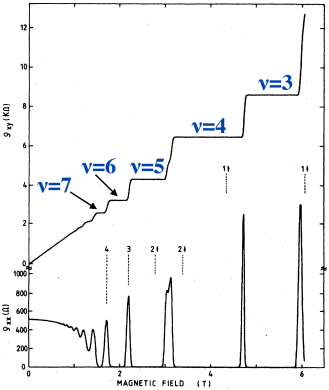
This seems to require $\nu \in 2\mathbb{Z}$,
but odd ν has been experimentally observed.

The resolution: **electrons are spinors**,
and therefore M_3, W_4 etc. require **spin structure**.

The intersection form on a spin 4-manifold is **even**,
and therefore $\nu \in \mathbb{Z}$.

This argument was implicitly known for a long time since late 80s, but the crucial factor of two related to spin structure was not appreciated very much until around 2000.

I think it is quite amazing that we see this fact experimentally in



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Aside: why is the intersection form even on a spin 4-manifold M?

It suffices to show that $\int_M x^2 = 0 \in \mathbb{Z}/2$ for any $x \in H^2(M, \mathbb{Z}/2)$.

This is because

$$\int_M x^2 \overset{(1)}{=} \int_M \mathrm{Sq}^2 x \overset{(2)}{=} \int_M v_2 x \overset{(3)}{=} \int_M (w_2 + w_1^2) x \overset{(4)}{=} 0.$$

(see e.g. Milnor-Stasheff (1974))

Modern times

2010s

Integer quantum Hall system is an example of

$(n+1)$-dimensional quantum field theory (QFT)
with **unique gapped ground state** with G-symmetry.

Often called

SPT phases

and/or

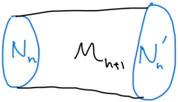
invertible phases.

(SPT= symmetry protected topological)

A more general $(n+1)$-dimensional quantum field theory (QFT) Q assigns a Hilbert space to a spatial manifold N_n:

$$N_n \mapsto \mathcal{H}_Q(N_n),$$

and for



it assigns

$$Z_Q(M_{n+1}) : \mathcal{H}_Q(N_n) \to \mathcal{H}_Q(N_n').$$

The manifold can be equipped with various structures of your choice, orientation, spin structure, G-bundle with connection, etc., giving rise to different flavors of QFTs.

We assume $\mathcal{H}_Q(\varnothing) = \mathbb{C}$, then

$$Z_Q(M_{n+1}) : \mathcal{H}_Q(\varnothing) \to \mathcal{H}_Q(\varnothing)$$

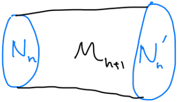
determines a complex number

$$Z_Q(M_{n+1}) \in \mathbb{C},$$

called the partition function.

A QFT Q is **SPT/invertible/with unique gapped ground state** $\Leftrightarrow \mathcal{H}_Q(N)$ **is always 1-dimensional**.

Integer quantum Hall material is a $(2+1)$-dimensional **spin invertible** QFT with $U(1)$ **symmetry**:

$Z_Q($ 

$) : \mathcal{H}_Q(N) \to \mathcal{H}_Q(N')$

N, N' are 2-dimensional; M is 3-dimensional;
they come with **spin** structure and $U(1)$ **bundle with connection**,

and $\mathcal{H}_Q(N)$ is **always 1-dimensional**.

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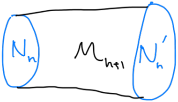
$Z_Q($ 

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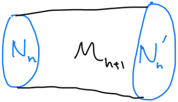
$Z_Q($ 

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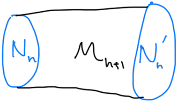
$Z_Q($ 

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We would like to understand

$$\mathbf{Inv}^{n+1}_{\mathcal{S},G} := \pi_0(\{\begin{array}{c}(n+1)\text{-dim. } \textbf{invertible} \text{ QFTs} \\ \text{with structure } \mathcal{S} \text{ and symmetry } G\end{array}\})$$

Here $\mathcal{S}$ can be spin structure, orientation only, etc.

As invertible QFTs form a group under tensor product

$$\begin{aligned}\mathcal{H}_{Q\times Q'}(N) &= \mathcal{H}_Q(N) \otimes \mathcal{H}_{Q'}(N),\\ Z_{Q\times Q'}(M) &= Z_Q(M) \otimes Z_{Q'}(M), \quad \text{etc.,}\end{aligned}$$

$\mathbf{Inv}^{n+1}_{\mathcal{S},G}$ will be an Abelian group.

Dijkgraaf-Witten (1990)

$$\mathbf{Inv}^{n+1}_{?,G} \overset{\text{proposal}}{=} H^{n+2}(BG,\mathbb{Z})$$

Dependence on $\mathcal{S}$ not appreciated at that time.
Wrong if taken too literally.

Integer quantum Hall effect is the case $n = 2$, $G = U(1)$. Then

$$H^4(BU(1),\mathbb{Z}) \simeq \mathbb{Z}$$

is generated by $(c_1)^2$, but we need $\frac{1}{2}(c_1)^2$ as we saw,
for which the spin structure was crucial.

Chen-Gu-Liu-Wen (2011)

$$\mathbf{Inv}^{n+1}_{\text{oriented},G} \overset{\text{proposal}}{=} H^{n+2}(BG, \mathbb{Z})$$

An influential paper, which introduced
and popularized the notion of SPT phases.

(The terminology "invertible phases" originates
from Freed-Moore (2004).)

Now known to be wrong for $n \geq 4$.

How about the **spin** case?

Freed (2006), Gu-Wen (2012)

$$\mathbf{Inv}^{n+1}_{\mathsf{spin},G} \overset{\text{proposal}}{=} E^{n+2}(BG)$$

where E^d is a cohomology theory given by

$$E^d(X) = \frac{\left\{(a,b) \in C^{d-3}(X,\mathbb{Z}/2) \times C^d(X,\mathbb{Z}) \;\middle|\; \begin{array}{l} \delta a = 0, \\ \delta b = \beta \circ \mathrm{Sq}^2\, a \end{array} \right\}}{\text{certain equiv. relation}}$$

where

β is the Bockstein for $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$ and
Sq^2 is the Steenrod square.

(Amazingly, Gu and Wen rediscovered
the cochain-level expression of Sq^2 by themselves!)

Schnyder-Ryu-Furusaki-Ludwig (2008), Kitaev (2009)

$$KO^{n-2}(pt) \to \mathrm{Inv}^{n+1}_{\mathsf{spin},pt}$$

They classified **free** **spin invertible phases without additional symmetry**.

They also considered structures related but not quite **spin**
(such as imposing time reversal, corresponding to considering **pin**$\pm$)
so that the classification is $KO^{n+i}(pt)$ for arbitrary i mod 8.

Called the **periodic table** of **free** **topological superconductors**.

(see e.g. a nice lecture by Ryu)

Kitaev (2015)

$$\mathbf{Inv}_{\mathcal{S},G}^{n+1} = E_{\mathcal{S}}^{n+2}(BG)$$

where $E_{\mathcal{S}}$ **should be a generalized cohomology theory.**

Kitaev only gave a talk and never wrote it up.

Fleshed out in Xiong, (2017) and Gaiotto, Johnson-Freyd (2017) etc.

Kapustin-Thorngren-Turzillo-Wang (2014)
Freed-Hopkins (2016)

$$\mathbf{Inv}^{n+1}_{\mathcal{S},G} \overset{\text{accepted}}{=} (D\Omega^{\mathcal{S}})^{n+2}(BG)$$

where $\Omega^{\mathcal{S}}$ is the $\mathcal{S}$-bordism homology and D is the Anderson dual.

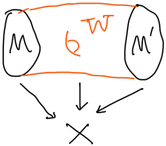
A generalized (co)homology theory $h^n(X)$, $h_n(X)$ satisfies the Eilenberg-Steenrod axioms for the ordinary (co)homology **except** the dimension axiom.

So $h_n(pt) = h^{-n}(pt)$ can be nontrivial for $n \neq 0$.

Bordism group

$$\Omega_n^{\mathcal{S}}(X) = \left\{ \begin{array}{c} \mathcal{S}\text{-structured manifold } M_n \\ \text{together with } f : M_n \to X \end{array} \right\} \Big/ \text{ bordism}$$

is an example, where

$M \overset{\text{bordant}}{\sim} M' \quad \Leftrightarrow$ 

For a generalized homology theory $h_*(-)$,
there is the **Anderson dual** cohomology theory $Dh^*(-)$
which satisfies the analogue of the universal coefficient theorem:

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(h_{d-1}(X), \mathbb{Z}) \\ \to (Dh)^d(X) \to \\ \mathrm{Hom}_{\mathbb{Z}}(h_d(X), \mathbb{Z}) \to 0$$

The universal coefficient theorem of $H(-, \mathbb{Z})$ means that

$$DH(-, \mathbb{Z}) = H(-, \mathbb{Z}).$$

Similarly, $DK = K$ and $DKO^\bullet = KO^{\bullet+4}$.

Classification of fermionic invertible phases

$$\mathbf{Inv}^{n+1}_{\mathsf{spin},G} \overset{\text{accepted}}{=} (D\Omega^{\mathsf{spin}})^{n+2}(BG)$$

$\Omega^{\mathsf{spin}}_{\bullet}(pt)$ was determined in Anderson-Brown-Peterson (1967) and the Anderson dual was introduced in Anderson (1969).

Physicists now need them!

That's why graduate students in condensed matter physics learn the Atiyah-Hirzebruch spectral sequence and the Adams spectral sequence to compute them.

Present

2020s

The last topic of the talk is about **physics and elliptic cohomology**.

There are three types of complex curves with Abelian group law:

$$\mathbb{C}, \quad \mathbb{C}^\times, \quad \text{elliptic curves.}$$

Correspondingly, there are three types of cohomology theories:

$$H^*(-,\mathbb{Z}), \quad K^*(-), \quad \text{elliptic cohomologies.}$$

They are all complex orientable: a complex n-fold M_{2n} has the fundamental class $[M_{2n}] \in E_{2n}(M)$.

All these cohomology theories have the 1st Chern class $c_1(\mathcal{L}) \in E^*(X)$ for complex line bundles $\mathcal{L} \to X$.

The group law dictates how $c_1(\mathcal{L} \otimes \mathcal{L}')$ is expressed in terms of $c_1(\mathcal{L})$ and $c_1(\mathcal{L}')$.

Today I would like to discuss their real analogues:

$$\boldsymbol{H^*(-,\mathbb{Z}),\quad KO^*(-),\quad TMF^*(-).}$$

$\boldsymbol{TMF}$ is the topological modular form, constructed by Hopkins et al. in late 1990s. (cf. Hopkins' talk at ICM 2002)

I hear the construction uses a sheaf of $\boldsymbol{E_\infty}$-ring specta over the moduli stack of elliptic curves over $\mathbb{Z}$.

I don't understand any of the words in the last sentence.

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M_n has a fundamental class in $TMF_n(M)$
if M is **string**. $=$ the trivialization of $p_1(TM)$ is given.

Note that the first three nontrivial homotopy group of O is

$$\pi_0(O) = \mathbb{Z}/2, \quad \pi_1(O) = \mathbb{Z}/2, \quad \pi_3(O) = \mathbb{Z}$$

and w_1, w_2, p_1 are the corresponding obstruction classes.

Adams spectral sequences computing them have the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(0)}(H^*(X,\mathbb{Z}/2),\mathbb{Z}/2) \Rightarrow H_{t-s}(X,\mathbb{Z})_{\hat{2}}$$
$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*(X,\mathbb{Z}/2),\mathbb{Z}/2) \Rightarrow ko_{t-s}(X)_{\hat{2}}$$
$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(H^*(X,\mathbb{Z}/2),\mathbb{Z}/2) \Rightarrow tmf_{t-s}(X)_{\hat{2}}$$

where $\mathcal{A}(n)$ is the subalgebra of the Steenrod algebra generated by Sq^1, Sq^2, ..., Sq^{2^n}.

TMF is the natural next entry after $H(-,\mathbb{Z})$ and KO.

$\boldsymbol{KO}$ is $\mathbf{8}$-periodic:

$$\boldsymbol{KO^{n+8}(X) \simeq KO^n(X)}$$

$\boldsymbol{TMF}$ is $\mathbf{24^2 = 576}$-periodic:

$$\boldsymbol{TMF^{n+576}(X) \simeq TMF^n(X)}$$

TMF is called the topological **modular form** since there is a homomorphism

$$TMF_* \to MF_*[\Delta^{-1}]$$

where

$$MF = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta).$$

is the ring of integral modular forms, with

$$c_4 = 1 + 240q + \cdots, \quad c_6 = 1 - 504q - \cdots$$

are the Eisenstein series and

$$\Delta = q - 24q^2 + \cdots$$

is the modular disciminant.

$\boldsymbol{TMF_* \to MF_*[\Delta^{-1}]}$ is rationally isomorphic

$$\boldsymbol{TMF_* \otimes \mathbb{Q} \simeq MF_*[\Delta^{-1}] \otimes \mathbb{Q},}$$

and it is isomorphic at degree 0

$$\boldsymbol{TMF_0 = \mathbb{Z}[J]}$$

where $\boldsymbol{J}$ is the modular $\boldsymbol{J}$-invariant,
but not surjective in general.

For example, $\boldsymbol{k\Delta}$ is in the image only when $\boldsymbol{24}$ divides $\boldsymbol{k}$.

$\boldsymbol{TMF_* \to MF_*[\Delta^{-1}]}$ also has a lot of torsion.

$\boldsymbol{KO^n(X)}$ has a geometric realization: for $\boldsymbol{n = 0}$,
it is given by virtual differences of real vector bundles over $\boldsymbol{X}$.

Is there a similarly nice realization of $\boldsymbol{TMF^n(X)}$?

$KO^n(X)$ has a geometric realization: for $n = 0$,
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Is there a similarly nice realization of $TMF^n(X)$?

Segal-Stolz-Teichner conjecture

$$TMF^n(X) = \pi_0 \left\{ \begin{array}{c} 2\text{-dim'l supersymmetric QFT} \\ \text{of degree } n \text{ parameterized by } X \end{array} \right\}$$

Segal 1988, Stolz-Teichner 2002, 2011

This is a very difficult conjecture. The RHS isn't even defined yet.

An easier version is:

$$KO^n(X) = \pi_0 \left\{ \begin{array}{c} \mathbf{1}\text{-dim'l } \textbf{time-reversal invariant} \\ \textbf{supersymmetric} \text{ QFT} \\ \text{of } \textbf{degree } n \text{ parameterized by } X \end{array} \right\}$$

which was rigorously formulated and proved.

Roughly: a **1**-dim'l supersymmetric QFT is just
a supersymmetric quantum mechanics, and

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Degree n means that there is an action of $Cl(n, \mathbb{R})$.

Therefore the statement becomes

$$KO^n(X) \overset{?}{=} \pi_0 \left\{ \begin{array}{c} \text{family of } \textbf{odd self-adjoint operators } Q \\ \text{parameterized over } X \\ \text{on a } \mathbb{Z}/2\textbf{-graded real } \text{Hilbert space } \mathcal{H} \\ \text{commuting } \textbf{with } Cl(n, \mathbb{R}) \textbf{ action} \end{array} \right\}$$

and the RHS is more or less the definition of KO in terms of Fredholm operators.

(For a detailed proof, see e.g. Cheung 2008.)

In this description, the pushforward

$$\pi_! : KO^0(M) \to KO^{-n}(pt)$$

of $[V] \in KO^0(M)$ for an n-dimensional M is given by

$$\mathcal{H} = \Gamma(V \otimes S(\mathbb{R}^{8k-n} \oplus TM)),$$
$$Q = \text{Dirac operator on it.}$$

The TMF version is much harder:

$$\boldsymbol{TMF^n(X) = \pi_0}\left\{ \begin{array}{c} \text{2-dim'l supersymmetric QFT} \\ \text{of degree } \boldsymbol{n} \text{ parameterized by } \boldsymbol{X} \end{array} \right\}$$

The LHS involves sheaves of spectra over the moduli stack of elliptic curves over $\mathbb{Z}$.

The RHS involves QFTs, which seem to me a purely characteristic-0 phenomenon.

Still, nontrivial physics motivation and checks.

For example, take

$$\boldsymbol{TMF_3(pt) = \mathbb{Z}/24,}$$

which is naturally isomorphic to

$$\boldsymbol{\Omega_3^{\text{framed}}(pt) = \pi_3^S(pt) = \lim \pi_{n+3} S^n.}$$

In the standard math definition, the computation involves elliptic curves in characteristic $\mathbf{2}$ and $\mathbf{3}$.

The same $\mathbb{Z}/\mathbf{24}$ also follows from an intricate construction in QFT.

Gaiotto, Johnson-Freyd 2019

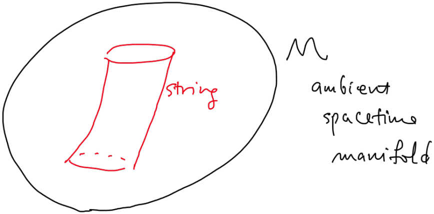
Historically, elliptic cohomologies / TMF came from two strands of ideas.

One is purely from within algebraic topology, called chromatic phenomena, about which I have no clue.

Another is from Witten.

(This part of the story is nicely summarized in Landweber 1988.)

In string theory we consider strings moving in a manifold:



This should be described by a **2-dim'l supersymmetric QFT** on the worldsheet of the string.

It gives rise to a **sequence of Dirac operators** acting on the spinor bundle SM tensored with tensor powers of the tangent bundle TM.

In 1984, Witten asked the property of the index of these operators to Landweber and Stong, who then informed Ochanine about the question.

By 1986, they realized that there is a generalization of the $\hat{A}$ genus

$$\int_M \hat{A} \in \mathbb{Z}$$

which takes the values in modular forms

$$\int_M \phi_W \in MF.$$

Here, M needs to be spin (i.e. $w_2 = 0$) for the former
and string (i.e. $p_1 = 0$) for the latter.

$\hat{A}$ was known to come from KO.
There should be some nice cohomology theory for ϕ_W.
It took about 15 years for mathematicians to construct TMF.

But physicists were almost completely detached from these developments until very recently.

Only in November 2018 papers on this topic appeared (by Gaiotto, Johnson-Freyd and Gukov-Pei-Putrov-Vafa), in which **some physics checks of the Segal-Stolz-Teichner conjecture** were made.

Instead, **assuming the Segal-Stolz-Teichner conjecture**, we can use the known properties of $\boldsymbol{TMF}$ to **deduce the properties** of 2d supersymmetric QFTs and **of string theory**.

I wrote a short letter about it a few months ago;
and I am trying to generalize it further,
with the help of Yamashita at RIMS.

But the details need to be left to some other time. some more detail

Today I surveyed the interaction between physics and algebraic topology.

Concrete homotopy groups are useful in studying topological solitons.

(math: 1930s, physics: 1970s)

Anderson duals of bordism homologies classify SPT phases.

(math: 1960s, physics: 2010s)

TMF and 2d supersymmetric field theories

(math: 2000s, physics: 2020s)

We're trailing behind, but slowly catching up.

Back-up slides

Excerpt from Wigner

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XIII, 001–14 (1960)

The Unreasonable Effectiveness of Mathematics in the Natural Sciences

Richard Courant Lecture in Mathematical Sciences delivered at New York University, May 11, 1959

EUGENE P. WIGNER
Princeton University

"and it is probable that there is some secret here which remains to be discovered." (*C. S. Peirce*)

There is a story about two friends, who were classmates in high school, talking about their jobs. One of them became a statistician and was working on population trends. He showed a reprint to his former classmate. The reprint started, as usual, with the Gaussian distribution and the statistician explained to his former classmate the meaning of the symbols for the actual population, for the average population, and so on. His classmate was a bit incredulous and was not quite sure whether the statistician was pulling his leg. "How can you know that?" was his query. "And what is this symbol here?" "Oh," said the statistician, "this is π." "What is that?" "The ratio of the circumference of the circle to its diameter." "Well, now you are pushing your joke too far," said the classmate, "surely the population has nothing to do with the circumference of the circle."

https://doi.org/10.1002/cpa.3160130102

Mathematical Logic and Physics?

This page is just to keep a paper I recently got to know.

There is a translation-invariant local Hamiltonian on a 1d lattice which is gapless/gapped only when ZFC is inconsistent/consistent.

[Cubitt, 2105.09854]

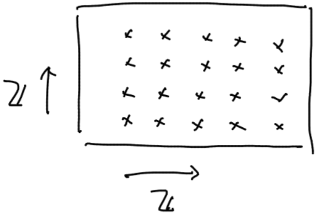
The point is to combine two techniques:

- There is a known way to encode a Turing machine to a translation-invariant local Hamiltonian on a 1d lattice so that it is gapless/gapped only when the said Turing machine halts or not.
  [Baush, Cubitt, Lucia, Perez-Garcia 1810.01858]
- There is a Turing machine which does not halt if and only if ZFC is consistent. [Yedidia, Aaronson 1605.04343].

back

Microscopic understanding of integer quantum Hall effect

It starts from the lattice structure in two-dimensional material:



Therefore

$$\mathbb{Z}^2 \curvearrowright \mathcal{H}$$

which allows us to decompose $\mathcal{H}$ in terms of the character

$$T^2 = \mathrm{Hom}(\mathbb{Z}^2, U(1)).$$

This means that $\mathcal{H}$ is the space of sections

$$\psi : T^2 \to \mathcal{H}'$$

of a trivial Hilbert space bundle

$$T^2 \times \mathcal{H}'$$

and the Hamiltonian H has the form

$$(H\psi)(p) = h(p)(\psi(p)) \qquad p \in T^2$$

where $h(p) : \mathcal{H}' \to \mathcal{H}'$ is a self-adjoint operator.

The gapped condition says that the lowest eigenvalue of $h(p)$ is non-degenerate, which determines a one-dimensional subspace

$$L(p) \subset \mathcal{H}'.$$

It forms a line bundle $\mathcal{L} \to T^2$ which is a sub-bundle of $T^2 \times \mathcal{H}'$.

A standard computation using the Kubo formula says that the Hall conductivity is

$$\sigma_H = \frac{e^2}{h} \int_{T^2} c_1(\mathcal{L})$$

and therefore it is an integer multiple of e^2/h.

Thouless-Kohmoto-Nightingale-den Nijs (1982)

back

Anomalies of heterotic string theories

What is an anomaly?

I said that an n-dim'l QFT Q assigns the partition function

$$Z_Q(M_n) \in \mathbb{C},$$

but the partition function of an anomalous QFT Q is instead given as

$$Z_Q(M_n) \in \mathcal{H}_{\mathcal{A}}(M)$$

where $\mathcal{A}$ is an $(n+1)$-dim'l **invertible** QFT and
$\mathcal{H}_{\mathcal{A}}$ is its Hilbert space which is one dimensional.

There are many anomalous QFTs.
Notable examples are free massless fermions, for which $\mathcal{H}_{\mathcal{A}}(M)$ is the determinant line bundle of the Dirac operator.

A n-dim'l possibly-anomalous spin QFT Q has

$\mathcal{A}$: a $(n+1)$-dim'l spin invertible QFT

as part of the data.

This is given by an element

$$\mathcal{A} \in \mathbf{Inv}^{n+1}_{\mathrm{spin}} = (D\Omega^{\mathrm{spin}})^{n+2}.$$

Now, there is a procedure called the **second quantization** you learn in the basic QFT course.

This is a machinery which does

$$\{\text{time-reversal-invariant quantum mechanics of degree } n-2\}$$
$$\downarrow$$
$$\{\text{possibly-anomalous } n\text{-dim'l spin QFT }\}$$

Applying the Stolz-Teichner for the source and the anomaly for the target, we have a homomorphism

$$KO^{n-2} \to (D\Omega^{\text{spin}})^{n+2}.$$

This is the Anderson dual to the spin orientation of the KO theory:

$$\Omega^{\text{spin}} \to KO^n$$

where we use $DKO^{n+4} = KO^n$.

My interest is the anomaly of heterotic string theory, which is a machinery which does

$$\{\text{2-dim'l supersymmetric QFT of degree } \boldsymbol{n+22}\}$$
$$\downarrow$$
$$\{\text{possibly-anomalous } \boldsymbol{n}\text{-dim'l quantum gravity with string structure }\}$$

Again applying the Stolz-Teichner for the source and the anomaly for the target, we have a homomorphism

$$\boldsymbol{TMF^{n+22} \to (D\Omega^{\text{string}})^{n+2}}.$$

String theory is often non-anomalous from miraculous reasons.
So we would like to know whether this homomorphism is zero.

$$TMF^{n+22} \to (D\Omega^{\mathrm{string}})^{n+2}$$

The seminal paper of Green and Schwarz (1984), which started superstring theory as we know it,
showed that the image of a certain element of TMF^{10+22} is torsion.

The paper by Witten with an appendix by Strong (1986) proved that the image of this particular element is actually zero.

Lerche-Nilsson-Schellekens-Warner (1988) showed that the image in general is torsion.

$$TMF^{n+22} \to (D\Omega^{\text{string}})^{n+2}$$

In my recent paper (2021), I showed that the map is trivial when $n = 2$, for which

$$(D\Omega^{\text{string}})^{n+2} = \mathrm{Hom}(\Omega_3^{\text{string}}, U(1)) = \mathbb{Z}_{24},$$

using a result of Hopkins 2002.

In an ongoing collaboration with Yamashita,
we show that the map is zero in general.

back