Quantum Anomalies
— Old and New —

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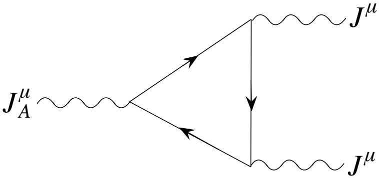
Adler-Bell-Jackiw anomaly (1969)

Consider

- a Dirac fermion ψ,
- its standard $\mathrm{U}(1)$ current $J^\mu = \bar{\psi}\gamma^\mu\psi$,
- and the axial $\mathrm{U}(1)$ current $J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$.

Naively, both $\partial_\mu J^\mu = \partial_\mu J_A^\mu = 0$.

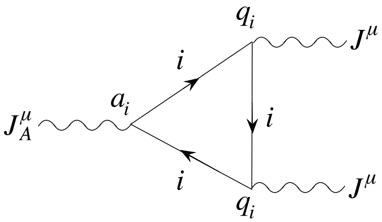
But a careful computation of



shows that they cannot be all conserved. We have

$$\partial_\mu J_A^\mu \propto \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

With many fermions, with ordinary $\mathbf{U(1)}$ charge q_i and an axial $\mathbf{U(1)}$ charge a_i, an easy generalization



gives

$$\partial_\mu J_A^\mu \propto \left[\sum_i a_i {q_i}^2\right] \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

Consider just u and d, and assign $a_u = +1$ and $a_d = -1$.

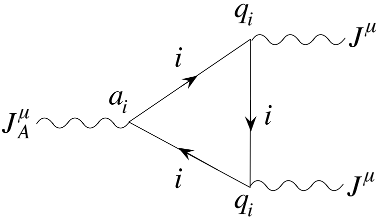
- This J_A^μ is spontaneously broken,
- and the corresponding NG boson is the neutral pion π^0,
- and there is also an explicit breaking term that gives a mass to π^0.

So J_A^μ creates a π^0:

$$\langle 0|J_A^\mu|\pi^0\rangle = \sqrt{2} i f_\pi p^\mu$$

where f_π is the pion decay constant.

And the process



whose amplitude is

$$\partial_\mu J_A^\mu \propto \left[\sum_i a_i {q_i}^2\right] \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

describes the decay $\pi^0 \to \gamma\gamma$.

Then the decay rate for $\pi^0 \to \gamma\gamma$ is given by

$$\Gamma(\pi^0 \to \gamma\gamma) = \left[N_c((\frac{2}{3})^2 - (\frac{1}{3})^2)\right]^2 \frac{\alpha^2}{32\pi^3}\frac{m_\pi^3}{f_\pi^2} \sim 7.7\text{eV}$$

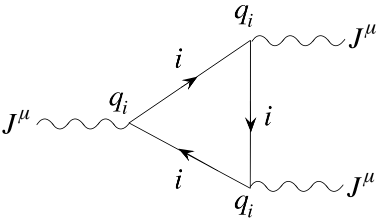
while the experimental value is $\sim 7.8\text{eV}$.

This was one of the earliest experimental checks of N_c; back then QCD was not established!

Gauge anomaly

Let us now change the convention from using **Dirac fermions** to **Weyl fermions**, every field regarded left-handed.

The anomaly is now



$\sim q_i{}^3.$

When J^μ is a gauge-current, we need to have

$$\sum_i q_i{}^3 = 0.$$

Otherwise the theory is inconsistent.

Take the Standard Model. A generation consists of

	Q_L	$\bar{u}_R$	$\bar{d}_R$	ℓ_L	$\bar{e}_R$
SU(3)	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$
SU(2)	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$
U(1)	$1/6$	$-2/3$	$1/3$	$-1/2$	1

Then we ask:

$$0 \stackrel{?}{=} 3 \cdot 2 \cdot \left(\frac{1}{6}\right)^3 + 3 \cdot \left(-\frac{2}{3}\right)^3 + 3 \cdot \left(\frac{1}{3}\right)^3 + 2 \cdot \left(-\frac{1}{2}\right)^3 + 1 \cdot 1^3$$

$$\mathbf{3 \cdot 2 \cdot \left(\frac{1}{6}\right)^3 + 3 \cdot \left(-\frac{2}{3}\right)^3 + 3 \cdot \left(\frac{1}{3}\right)^3 + 2 \cdot \left(-\frac{1}{2}\right)^3 + 1 \cdot 1^3}$$

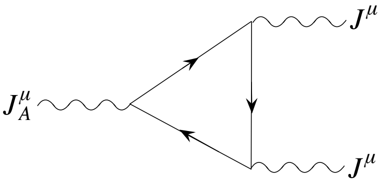
$$
\begin{aligned}
& 3\cdot 2\cdot\left(\frac{1}{6}\right)^3+3\cdot\left(-\frac{2}{3}\right)^3+3\cdot\left(\frac{1}{3}\right)^3+2\cdot\left(-\frac{1}{2}\right)^3+1\cdot 1^3 \\
= \;& \frac{1}{36}-\frac{8}{9}+\frac{1}{9}-\frac{1}{4}+1
\end{aligned}
$$

$$
\begin{aligned}
& 3 \cdot 2 \cdot \left(\frac{1}{6}\right)^3 + 3 \cdot \left(-\frac{2}{3}\right)^3 + 3 \cdot \left(\frac{1}{3}\right)^3 + 2 \cdot \left(-\frac{1}{2}\right)^3 + 1 \cdot 1^3 \\
= \; & \frac{1}{36} - \frac{8}{9} + \frac{1}{9} - \frac{1}{4} + 1 \\
= \; & \frac{1 - 32 + 4 - 9 + 36}{36}
\end{aligned}
$$

$$
\begin{aligned}
& 3\cdot 2\cdot\left(\frac{1}{6}\right)^3+3\cdot\left(-\frac{2}{3}\right)^3+3\cdot\left(\frac{1}{3}\right)^3+2\cdot\left(-\frac{1}{2}\right)^3+1\cdot 1^3\\
= & \frac{1}{36}-\frac{8}{9}+\frac{1}{9}-\frac{1}{4}+1\\
= & \frac{1-32+4-9+36}{36}\\
= & 0 \quad !!
\end{aligned}
$$

One-loop nature of the computation so far

So far we only used the one-loop diagram



One question

$\Gamma(\pi^0 \to \gamma\gamma)_{\text{one-loop}} \sim 7.7\text{eV}$ while $\Gamma(\pi^0 \to \gamma\gamma)_{\text{exp}} \sim 7.8\text{eV}$.

Why is the agreement so good?

It is, after all, the low energy QCD and very strongly coupled.

One-loop nature of the computation so far

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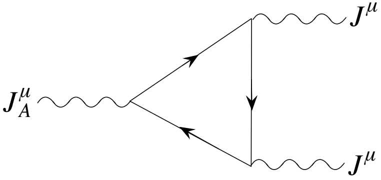

One worry

The one-loop gauge anomaly of the Standard Model is nicely cancelled.
How about the higher loops?

Non-renormalization of anomalies

In fact, it is known that the **anomaly receives no further corrections**, in the sense that in the equation

$$\partial_\mu J_A^\mu \propto e^2 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\nu\rho}$$

every further correction is absorbed in the renormalization of e.

Known under various names:

- the Adler-Bardeen theorem
- the 't Hooft anomaly matching

It was shown originally using various hard computations.

In the early 80s, people realized that
no computation was in fact necessary,
because if formulated in the right way,
the anomaly is an integer quantity.

Assuming that, the proof of the non-renormalization is easy:

1. The renormalization is a continuous process.

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2. You can't modify an integer in a continuous way.

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no computation was in fact necessary,
because if formulated in the right way,
the anomaly is an integer quantity.

Assuming that, the proof of the non-renormalization is easy:

1. The renormalization is a continuous process.
2. You can't modify an integer in a continuous way.
3. There's no third step.

To have the right formulation, we move from the canonical normalization

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \gamma^{\mu} (\partial_{\mu} + i e A_{\mu}) \psi$$

to the geometrical normalization

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \gamma^{\mu} (\partial_{\mu} + i 1 A_{\mu}) \psi.$$

I think in hep-th the geometrical normalization is by now the norm.

The geometrical normalization is so that around the monopole

$$\int F = \textbf{integer}$$

and around the instanton

$$\int \mathrm{tr}\, F \wedge F = \textbf{integer}$$

where I suddenly moved to the differential form notation.

It is just that every index is contracted against epsilon symbols.

Let

$$Z_\psi[A] = \int [D\psi][D\bar{\psi}] \exp\left[-\int d^4x \bar{\psi} i\sigma^\mu(\partial_\mu + iA_\mu)\psi\right].$$

The anomaly is the nontrivial gauge-dependence of the phase:

$$Z_\psi[A^g] = e^{i\varphi(g)} Z_\psi[A], \qquad A^g_\mu = gA_\mu g^{-1} + g\partial_\mu(g^{-1}).$$

We want to show that this anomalous phase is controlled by an integer.

One considers a **one-parameter family** of gauge transformations:

$$g(\theta) := g(\theta; x_0, x_1, x_2, x_3), \qquad 0 \le \theta \le 2\pi$$

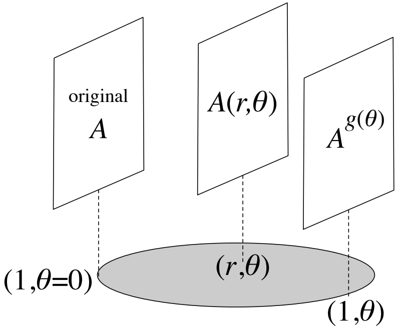
and $g(0) = g(2\pi)$. We then have a **one-parameter** family of gauge fields:

$$\begin{aligned} A_\mu(\theta; x_0, x_1, x_2, x_3) &:= A_\mu{}^{g(\theta)} \\ &= g(\theta) A_\mu g(\theta)^{-1} + g(\theta)\partial_\mu g(\theta)^{-1}. \end{aligned}$$

We now consider a **two-parameter family** $A_\mu(r, \theta; x_0, x_1, x_2, x_3)$ by introducing another parameter $0 \le r \le 1$:

$$A_\mu(r = 1, \theta) = A_\mu(\theta).$$

We now have the following situation:

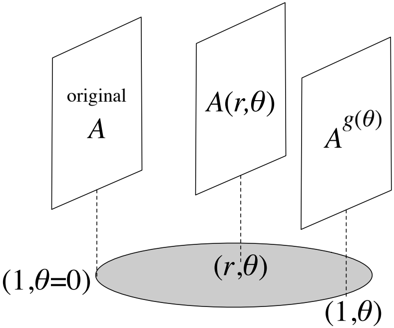


On the disk, we have a function $Z_\psi[A(r,\theta)]$, and around the boundary

$$Z_\psi[A(r=1,\theta)] = e^{i\varphi(g(\theta))} Z_\psi[A].$$

The phase $\varphi(g(\theta))$ can cover $0 \sim 2\pi n$ while $0 \leq \theta \leq 2\pi$.

We now have the following situation:



$A(r, \theta, x_0, x_1, x_2, x_3, x_4)$ can also be thought of as a gauge field in 6d.

$$\int \mathrm{tr}\, F \wedge F \wedge F = \textbf{integer}.$$

So we have two integers:

Around the boundary

$$Z_\psi[A(r=1,\theta)] = e^{i\varphi(g(\theta))} Z_\psi[A]$$

and the phase $\varphi(g(\theta))$ can cover $0 \sim 2\pi n$ while $0 \leq \theta \leq 2\pi$.

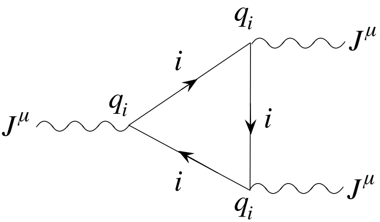
And as a 6d configuration,

$$\int \mathrm{tr}\, F \wedge F \wedge F = \textbf{integer}.$$

A computation shows that

$$n = k \int \mathrm{tr}\, F \wedge F \wedge F$$

where k is the strength of



.

In the equation

$$\boldsymbol{n} = \boldsymbol{k} \int \mathbf{tr}\, \boldsymbol{F} \wedge \boldsymbol{F} \wedge \boldsymbol{F}$$

- $\boldsymbol{n}$ measures the rotation number of the phase under a given family of gauge transformation
- $\mathbf{tr}\, \boldsymbol{F} \wedge \boldsymbol{F} \wedge \boldsymbol{F}$ is the topological number of the associated 6d gauge configuration
- $\boldsymbol{k}$ is the value of $\boldsymbol{n}$ when $\int \mathbf{tr}\, \boldsymbol{F} \wedge \boldsymbol{F} \wedge \boldsymbol{F} = \mathbf{1}$.

Can't be renormalized further!

Anomaly inflow

A related construction is as follows. For simplicity assume everything to be Abelian. Consider instead the Chern-Simons action

$$k \int A \wedge F \wedge F \sim k \int \epsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau}$$

in **five** dimensions.

Consider the gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$.

Then

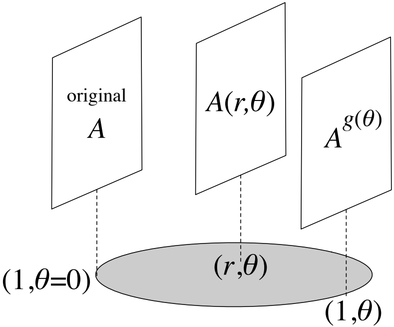
$$\delta \left[k \int_M A \wedge F \wedge F \right] = k \int_M d(\chi \wedge F \wedge F) = k \int_{\partial M} \chi \wedge F \wedge F.$$

Gauge-dependent only at the boundary.

In fact the relation

$$n = k \int F \wedge F \wedge F$$

where n is the gauge variation around a loop and F is the 6d configuration



guarantees that ...

The gauge variation

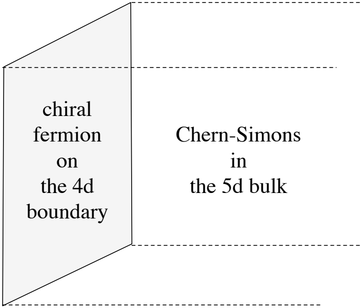
$$\delta\left[k\int_M A\wedge F\wedge F\right] = k\int_{\partial N}\chi\wedge F\wedge F$$

is exactly the gauge variation

$$Z_\psi[A^\chi] = e^{i\varphi(\chi)}Z_\psi[A]$$

of the chiral fermion on the boundary ∂M.

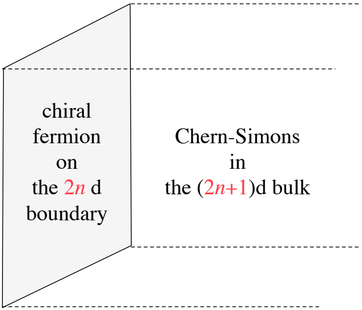
So the **total system**



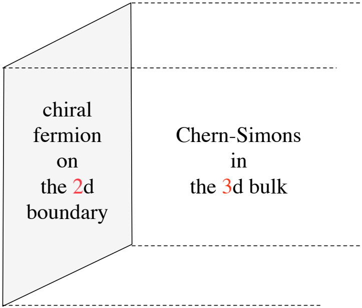
is gauge-invariant, since the gauge dependence of the bulk cancels against the gauge-dependence of the boundary.

[Faddeev-Shatashvili, Callan-Harvey 1985]

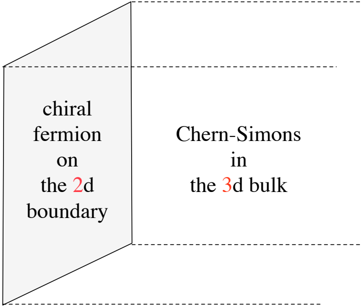
In fact the same derivation holds in any other dimensions



and in particular **between 2d and 3d**:



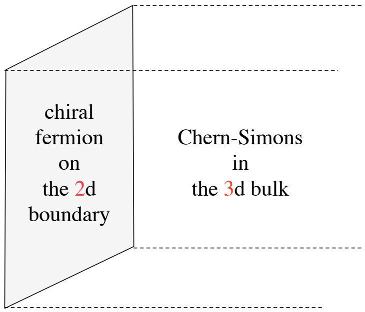
which has a condensed matter realization
called **integer quantum Hall effect.**



The bulk is $\mathbf{2+1}$d and the effective action is

$$k\int A\wedge F, \qquad k : \text{integer.}$$

Applying the Kubo formula we get quantized Hall conductance proportional to k, thus the name.



The boundary is $\mathbf{1+1}$d and there are massless chiral fermions there, known as the edge modes.

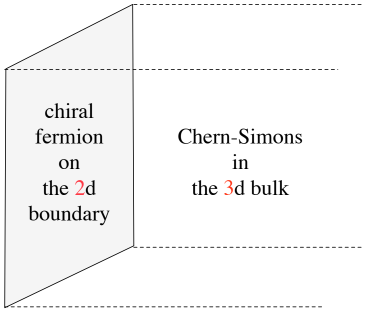
They are robust against experimental perturbation:
without them system is not gauge invariant.
So they can't be removed.

This understanding of the integer quantum Hall effect is commonplace by now, but I'm not sure who first formulated it this way and when.

The experimental discovery was in 1980
by [Von Klitzing, Dorda and Pepper].

By 1990 it was a common practice to use Chern-Simons to describe it.

So it should be sometime between the two.



The integer quantum Hall effect is the first example of **topological phases of matter** or equivalently **topological materials**.

Their essential feature is the bulk-boundary correspondence.

Let me explain next that **M-theory is also a topological material**.

Before doing that, recall the anomaly cancellation of String Theory.

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Before doing that, recall the anomaly cancellation of String Theory.

Consider any theory in 10d with supersymmetry. Suppose

- it contains gravity + superpartners
- it also contains gauge field + superpartners

Only two gauge groups, $\mathrm{SO}(32)$ or $E_8 \times E_8$ are allowed.

Otherwise it has a gauge anomaly and is inconsistent.

[Green-Schwarz 1984, Adams-Dewolfe-Taylor 2010]

Consider any theory in 11d with supersymmetry.
In fact it is believed there is only one, called M-theory.

There's no supermultiplet which contains a gauge field.

$g_{\mu\nu}$ has a superpartner

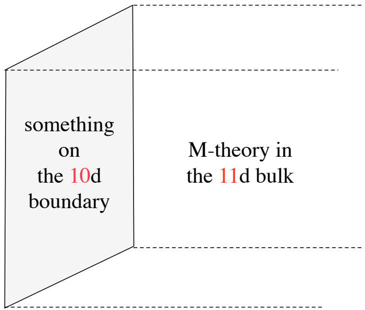
$$C_{[\mu\nu\rho]} \quad \text{with field strength} \quad G_{\mu\nu\rho\sigma} = \partial_{[\mu} C_{\nu\rho\sigma]}.$$

The SUSY forces a coupling

$$\propto \int_{11\mathrm{d}} \epsilon CGG.$$

Note that $3 + 4 + 4 = 11$.

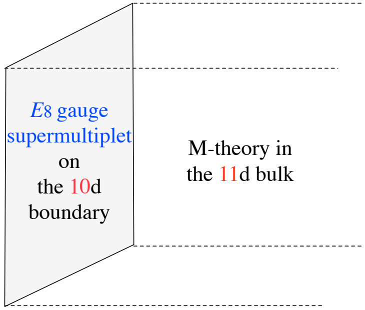
$\int_{11d} CGG$ has a nonzero gauge dependence at the boundary, just as $\int_{3d} AF$ did in the case of integer quantum Hall effect.



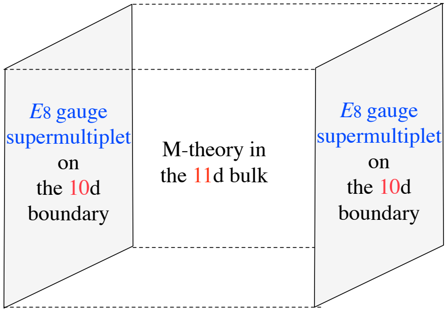
Suppose you can have a boundary on the 11d spacetime.

Then there needs to be something which is itself anomalous but cancels the bulk gauge dependence.

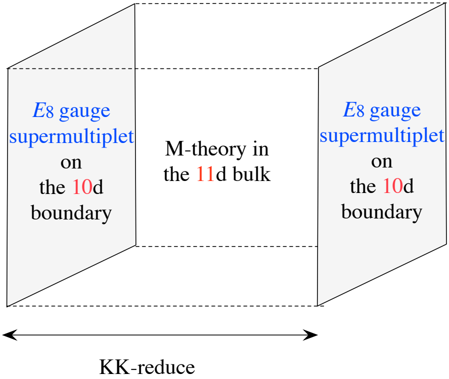
[Horava-Witten ’96] showed that a 10d E_8 gauge supermultiplet does the job:



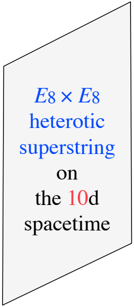
If you have two boundaries



We can Kaluza-Klein reduce along the horizontal direction



We have the $\boldsymbol{E_8} \times \boldsymbol{E_8}$ heterotic string



which is one of the two allowed choices of the gauge group in 10d.

So we have the analogue:

	integer quantum Hall material	:	chiral edge current
$\sim$	M-theory bulk	:	E_8 gauge multiplet

Quiz 1

Q: Is the anomaly only for fermions?

Quiz 1

Q: Is the anomaly only for fermions?

A: **No**. Can arise for self-dual bosons in 2d, 6d, 10d, ...

In 2d, for a scalar

$$\partial_\mu \phi = \epsilon_{\mu\nu} \partial^\nu \phi$$

In 6d, for a two-index antisymmetric field

$$\partial_{[\mu} B_{\nu\rho]} = \epsilon_{\mu\nu\rho\sigma\tau\upsilon} \partial^\sigma B_{\tau\upsilon}$$

etc.

Quiz 2

Q: Is the anomaly only in even dimensions?

Quiz 2

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A: **No**. E.g. in 5d, the spinor index of a fermion ψ_α has an antisymmetric invariant tensor $J_{[\alpha\beta]}$.

Then for a fermion ψ^a_α in the doublet of $\mathrm{SU}(2)$, one can impose

$$\psi^a_\alpha = J_{\alpha\beta}\epsilon^{ab}\bar{\psi}^\beta_b.$$

This half-doublet fermion has an anomaly.

Quiz 3

Q: Is the anomaly only for continuous symmetry?

Quiz 3

Q: Is the anomaly only for continuous symmetry?

A: **No**. Even a finite group symmetry (both internal and spacetime) can have an anomaly.

In the hep literature, the first paper I think was [Csaki-Murayama, 1997].

There is an independent series of works on the condensed matter side, since around 2005, under the name of

Symmetry Protected Topological (SPT) phases,

a sub-subject of topological phases of matter.

Anomaly of finite group symmetries

Let's consider the simplest case.

Let G be a finite group, and consider a 0+1 dimensional QFT.

Consider its partition function

$$Z(g) = \mathrm{tr}_{\mathcal{H}}\, \rho(g) e^{-\beta H}.$$

It is anomalous when $Z(g)$ is defined only up to a phase.

Examples?

Recall the concept of projective representations:

$$\rho(gh) = e^{i\varphi(g,h)}\rho(g)\rho(h).$$

The phase $\varphi(g,h)$ satisfies a certain consistency condition.

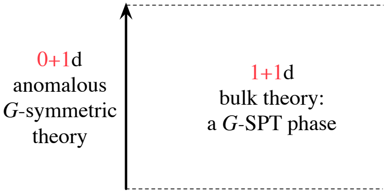
Then

$$\mathrm{tr}_{\mathcal{H}}\,\rho(gh)e^{-\beta H} = e^{i\varphi(g,h)}\,\mathrm{tr}_{\mathcal{H}}\,\rho(g)\rho(h)e^{-\beta H}.$$

G being anomalous in 0+1d QFT $\leftrightarrow$ G realized projectively on $\mathcal{H}$.

Note that **all states in $\mathcal{H}$ share the same phase $\varphi(g,h)$.**

The corresponding bulk theory

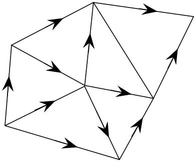


is called the G- symmetry protected topological (SPT) phase recently.

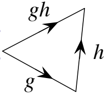
It's a 2d lattice theory whose action is given by $\varphi(g, h)$ appearing in

$$\rho(gh) = e^{i\varphi(g,h)}\rho(g)\rho(h).$$

One triangulates the 2d spacetime



and assign to each trianglette an element of the group, with the action

$S[$  $] = \varphi(g, h)$

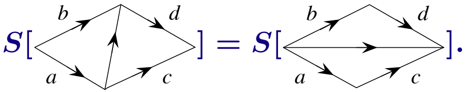
where, again,

$$\rho(gh) = e^{i\varphi(g,h)}\rho(g)\rho(h).$$

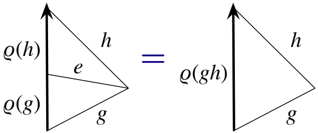
[Dijkgraaf-Witten, 1990]

$\varphi(g, h)$ is a phase in the projective representation $\Rightarrow$

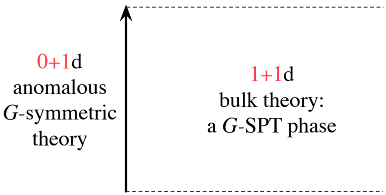
1. doesn't depend on how we triangulate the 2d spacetime since



2. cancels the gauge anomaly at the boundary since



So the bulk theory in



is this lattice theory, determined by the phase in

$$\rho(gh) = e^{i\varphi(g,h)}\rho(g)\rho(h).$$

This phase determines an element in $H^2(BG, \mathrm{U}(1))$.

In d spacetime dimensions, one instead needs $H^d(BG, \mathrm{U}(1))$ which also characterizes the anomaly of a finite group symmetry in $d-1$ dimensions. [X.G. Wen et al., 2011]

Another example: **the parity symmetry of a fermionic theory can have an anomaly** taking values in

d	$0+1$	$1+1$	$2+1$	$3+1$	$\cdots$
$P^2 = 1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	$\cdots$
$P^2 = (-1)^F$	$\mathbb{Z}_8$	0	0	0	$\cdots$

Yonekura-kun and I wrote a few paper on the entry $\mathbb{Z}_{16}$ last year.

There was a big competition among cond-mat people and us!

This idea of SPT phases is being imported back to the hep side.

For example, last month a paper by [Gaiotto, Kapustin, Komargodski and Seiberg] appeared, where the dynamics of non-supersymmetric pure Yang-Mills at $\theta = \pi$ was analyzed in detail using it.

You should have a journal club on it if you haven't.

Summary

I gave a very subjective overview of anomalies in the last 50 years.

- Adler-Bell-Jackiw anomaly and generalizations
- Non-renormalization
- Recent come-backs on the cond-mat side
- etc.

Thank you for your attention.