

Seiberg-Witten prepotential from Instanton Counting — A Review —

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1. Introduction

Seiberg-Witten theory

Seiberg and Witten determined the **exact** low energy Lagrangian for $d = 4, \mathcal{N} = 2$ $SU(N)$ SYM.

- VEVs of the adjoint scalar breaks $SU(N) \rightarrow U(1)^{N-1}$
- $\mathcal{N} = 2$ $U(1)^{N-1}$ theory is described by a single **holomorphic** function $\mathcal{F}(a_1, \dots, a_{N-1})$:

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left(\int d^4\theta \frac{\partial \mathcal{F}}{\partial a_i} a_i^\dagger + \int d^2\theta \frac{1}{2} \frac{\partial \mathcal{F}}{\partial a_i \partial a_j} W_i W_j \right)$$

- $\mathcal{F}$ can be determined by exploiting the duality symmetry.

Outcome: Take a **hyperelliptic**

$$y^2 = P(x)^2 - \Lambda^{2N}$$

with $\deg P(x) = N$ and take

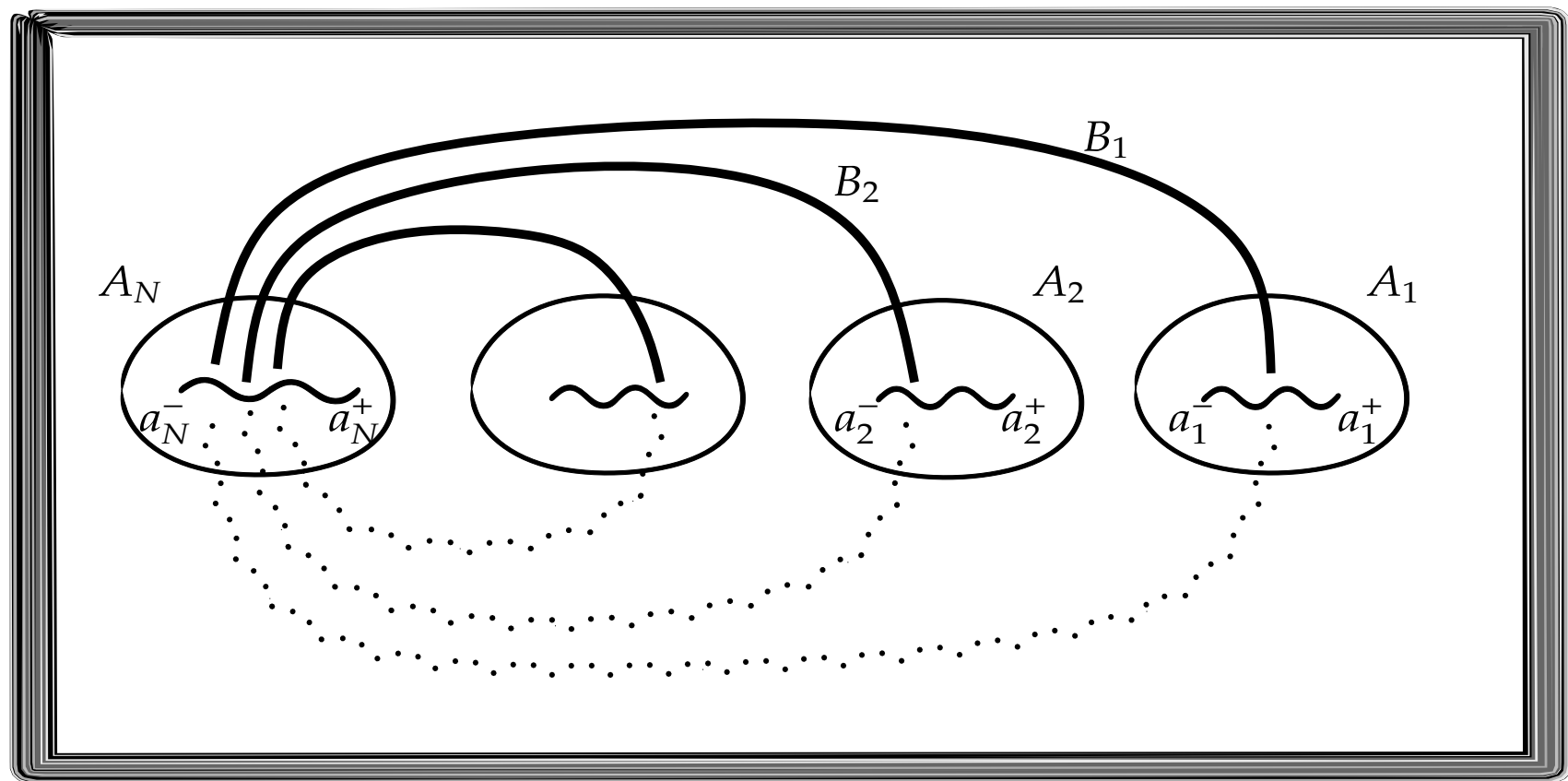
$$a_i = \oint_{A_i} dS, \quad a_D^i = \oint_{B^i} dS$$

where A_i and B^i form canonical bases of one cycles,

$$dS = x dP(x)/y : \quad \text{SW differential.}$$

Then, $\mathcal{F}$ satisfies

$$a_D^i = \frac{\partial \mathcal{F}}{\partial a_i}.$$



$\mathcal{F}$ is defined highly implicitly. After some work, we get

$$\begin{aligned}\mathcal{F} &= \frac{N}{\pi i} \sum a_k^2 && : \text{Bare term} \\ &+ \frac{i}{4\pi} \sum_{i < j} (a_i - a_j)^2 \log \frac{(a_i - a_j)^2}{\Lambda^2} && : \text{One-loop term} \\ &+ \sum_{m=1}^{\infty} \frac{\Lambda^{2Nm}}{2m\pi i} \mathcal{F}^{(m)}(a) && : \text{Instanton corrections}\end{aligned}$$

● From the RG eq. ,

$$\Lambda = \Lambda_0 e^{\frac{\pi i}{N} \left(\frac{\theta}{2\pi} + i \frac{4\pi}{g_0^2} \right)}, \quad \text{thus} \quad \Lambda^{2Nk} \sim e^{ik\theta}$$

which is appropriate for k -instanton contribution.

● Can we calculate $\mathcal{F}^{(m)}$ by instanton calculus?

Another Motivation

Consider type IIA on a CY

$X : \underbrace{\text{ALE singularity}}_{\text{Produces YM in 6d}} \text{ fibered over sphere}$

$\leadsto \mathcal{N} = 2 \text{ SYM in 4d.}$

Prepotential $\mathcal{F}_{g=0}$ for this theory can also be determined from the **world-sheet** point of view.

For this purpose, consider **topological A-model** on the same X :

$\mathcal{F}_{g=0}$ = genus zero free energy of this theory

Furthermore, from (Bershadsky-Cecotti-Ooguri-Vafa, 9309140), each term in

$$\log Z_{\text{topological theory}}^{(\text{all-genus})} = g_s^{-2} \mathcal{F}_{g=0} + \mathcal{F}_{g=1} + g_s^2 \mathcal{F}_{g=2} + \cdots$$

can be identified to the **graviphoton** correction to the prepotential:

$$\mathcal{F}_{\text{physical theory}}^{\text{graviphoton corrected}} = \mathcal{F}_{g=0} + \hbar^2 \mathcal{F}_{g=1} + \hbar^4 \mathcal{F}_{g=2} + \cdots$$

where $\hbar$: magnitude of the background graviphoton field strength.

We can now calculate $\mathcal{F}_g$ exactly, by a world-sheet technique called **topological vertex**. (Will be explained by Prof. Kanno's lecture tomorrow...)

If we can calculate $\mathcal{F}_g$ by means of spacetime instanton calculation and if the results agree, it should be a great consistency check for all these formalisms.

Instanton calculation:	(Nekrasov 0206161), (Y.T. 0401184)
Vertex Calculation & Checks for the equality:	(Iqbal-Kashani-Poor 0306032), (Eguchi-Kanno 0310235)

We review the achievements by Nekrasov in the following.

CONTENTS

- 1. Introduction (**DONE**)
- 2. Prepotential and Witten index of the instanton moduli
- 3. Fixed-Points in the instanton moduli and Young tableaux
- 4. Seiberg-Witten curves from Young tableaux
- 5. Conclusion

2. Prepotential and the Witten index of the moduli

Graviphoton-background

Firstly let us recall what's the graviphoton:

$$g_{\mu\nu} \text{ in 5d } \xrightarrow{S^1\text{-cptfy}} g_{\mu\nu} \text{ in 4d and } A_\mu = g_{5\mu}$$

Hence, **constant** background field strength for the graviphoton corresponds to the 5d geometry

$$ds^2 = (dx_\mu)^2 + (dx_5 + A_\mu dx_\mu)^2$$

with dA **constant**.

- S^1 is fibered over $\mathbb{R}^4$.
- Denote the eigenvalues of dA by $\pm \hbar$.

Gopakumar-Vafa invariants of 5d theory

When one puts 8-susy 5d theory on this background, the graviphoton-corrected prepotential is given by

$$\hbar^{-2} \mathcal{F} = \sum_{i,r} N_{i,r} \mathcal{F}_r(a_i) \quad \text{where} \quad \mathcal{F}_r(a) = \sum_{k>0} \frac{1}{k} \left(2 \sinh \frac{k\hbar}{2} \right)^{2r-2} e^{-ka}$$

and

$N_{i,r}$:# of ‘hypermultiplets’ with central charge a_i
and left spin content I_r .

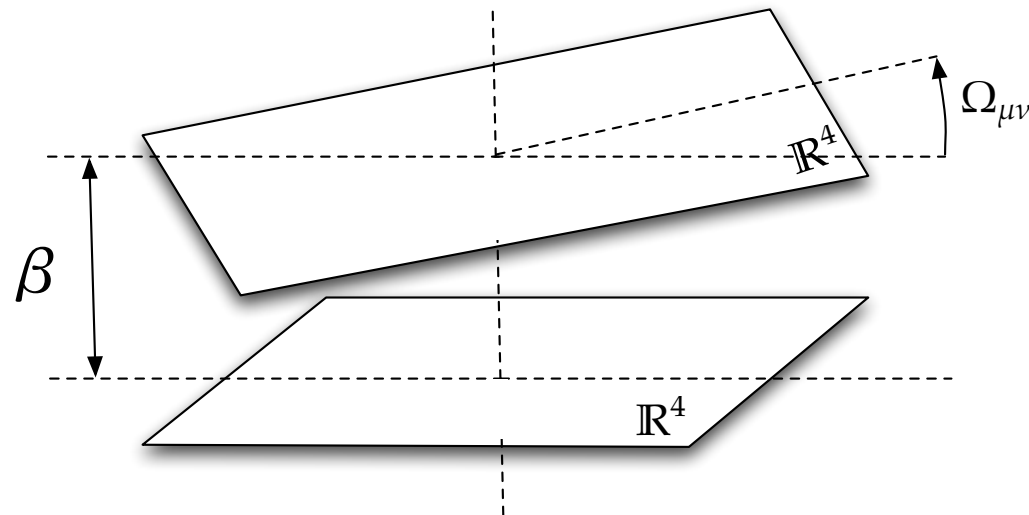
- $I_r = \left(\left(\frac{1}{2} \right) \oplus 2(0) \right)^r$
- Derived by free field calculation.

Ω -background

Consider another 5d background with metric

$$ds^2 = (dx^\mu + A_\mu dx^5)^2 + dx^5$$

with the same A_μ as before. $\mathbb{R}^4$ is fibered over S^1 .



Put the same 5d theory on this background.
The partition function should have the form

$$Z = \prod_{i,r} Z_r(a_i)^{N_{i,r}}.$$

Free field calculation shows that, surprisingly,

$$Z_r(a) = e^{-\mathcal{F}_r(a)}.$$

Hence,

$$Z_{\text{on } \Omega \text{ bkg.}} = \exp(-\hbar^{-2} \mathcal{F}_{\text{on graviphoton bkg.}})$$

i.e., in order to calculate $\mathcal{F}$, one can calculate $Z_{\text{on } \Omega}$ instead.

So, what's the Good News?

$Z_{\text{on } \Omega}$ can be expressed as

$$Z_{\text{on } \Omega} = \text{tr}(-)^F e^{-\beta H} e^{i\beta \hbar J}$$

by considering the 5th direction as 'time'.

None other than the **Witten Index**!

(J is one of the Lorentz generators.)

Application to the SYM

We have to encode the VEV of the adjoints.

We can accomplish this by introducing Wilson loops around the 5th direction, because

$$\text{4d adjoint scalar (complex)} = \text{5d adjoint scalar (real)} + \text{Wilson lines.}$$

Hence, the object to calculate is

$$e^{-\hbar^{-2}\mathcal{F}(a)} = Z_{\text{on } \Omega} = \text{tr}(-)^F e^{-\beta H} e^{i\beta\hbar J} e^{i\beta a_i J_i}$$

where J_i : generators of global gauge rotation.

Reduction onto the instanton moduli

Supersymmetries commuting $\hbar J + a_i J_i$ remains

$\implies$ only BPS configurations, i.e. **4d instantons** contribute.

For k -instanton configuration,

$$\text{energy} = k\tau.$$

Thus,

$$Z_{\text{on } \Omega} = \sum_{k \geq 0} e^{-\beta\tau k} \text{tr}(-)^F e^{-\beta H_k} e^{i\beta\hbar J} e^{i\beta a_i J_i}$$

Now H_k is the hamiltonian for

susy QM on the k -instanton moduli.

Fixed point theorem for the Witten index

In order to calculate $Z_{\text{on } \Omega}$, we utilize the
Atiyah-Bott-Lefschetz fixed point theorem.

(Setup)

- M : Spin manifold, $J \rightarrow M$: a $U(1)$ bundle on M
- an action of a group element $g = e^a$ on $J \rightarrow M$
- consider a susy QM with
its Hilbert space $= (\text{spin bundle of } M) \otimes J$
- We want to calculate

$$Z = \text{tr}(-)^F e^{-\beta H} e^a.$$

- This is β independent.

- In $\beta \rightarrow \infty$ limit, this calculates the character of the vacuum under g .
- In $\beta \rightarrow 0$ limit, centrifugal force restricts the contribution to that from the neighborhood of the **fixed points** of g . Evaluation of Gaussian fluctuation reveals

$$Z = \sum_{p:\text{f.p.}} e^w \prod_{\alpha=1}^{d/2} \frac{1}{e^{i\theta_\alpha/2} - e^{-i\theta_\alpha/2}}.$$

where

w : eigenvalue of a on $J|_p$
 $i\theta_\alpha$: eigenvalues of a on $TM|_p$.

An Example

- Take $M = S^2$ and J : charge n monopole configuration
- Consider $g = e^{i\theta}$: rotating around the z direction
- The vacua forms spin n representation of $SU(2)$

$$\leadsto Z = e^{in\theta} + e^{i(n-1)\theta} + \dots + e^{-in\theta}$$

- The fixed point theorem tells, as the fixed points are north and south poles,

$$Z = \frac{e^{i(n+1/2)\theta}}{e^{i\theta/2} - e^{-i\theta/2}} + \frac{e^{-i(n+1/2)\theta}}{e^{-i\theta/2} - e^{i\theta/2}}$$

They Agree!

Summary of this (rather lengthy) chapter

- $e^{-\hbar^2 \mathcal{F}(a)} = Z_{\text{on } \Omega}.$
- $Z_{\text{on } \Omega} = \sum_{k \geq 0} (\text{Witten index of the } k\text{-instanton moduli}).$
- $(\text{Witten index}) = \sum_{\text{f.p.}} (\text{contrib. from each f.p.}).$

The calculation of the prepotential reduced to the study of the fixed points in the instanton moduli!

3. Fixed points in the Instanton Moduli

ADHM construction of the Instanton Moduli

Consider the following system of branes:

	0	1	2	3	4	5	6	7	8	9
k D3's	—	—	—	—	●	●	●	●	●	●
N D7's	—	—	—	—	—	—	—	—	●	●

$\mathcal{N} = 2$ field theory on D3 is described by

$$\begin{aligned} \mathcal{L} = & \int d^4\theta (\Phi^\dagger \Phi + B_1^\dagger e^V B_1 + B_2^\dagger e^{-V} B_2) \\ & + \int d^2\theta \operatorname{tr}(I\Phi J) + \int d^2\theta \operatorname{tr}(B_1[\Phi, B_2]) + \int d^2\theta \operatorname{tr} W_\alpha W^\alpha + c.c. \end{aligned}$$

where

$$\begin{aligned}\Phi: & N \times N, & \text{position of D3s along 8,9} \\ B_1, B_2: & N \times N, & \text{position of D3s along 4,5,6,7} \\ I, J^\dagger: & N \times k, & \text{from D3-D7 strings}\end{aligned}$$

The susy vacua describes

Coulomb phase: D3's and D7's are separated along 8,9

Higgs phase: D3's are absorbed on the D7s as **instantons**
in 4,5,6,7 directions

$\Downarrow$

$$U(N) \text{ } k\text{-Instanton moduli} \sim \frac{\left\{ \begin{aligned} [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J &= 0 \\ [B_1, B_2] + IJ &= 0. \end{aligned} \right\}}{U(k) \text{ gauge invariance}}$$

Naïve analysis of the fixed points

- Eigenvalues of B_i are the centers of instantons
- They are rotated by $J \rightsquigarrow$ All centers should be **at the origin**
- Extremely singular configurations necessitates some regularization

$\Downarrow$

$$\begin{array}{c}
 U(N) \text{ non-commutative} \\
 k\text{-Instanton moduli}
 \end{array}
 \sim
 \frac{
 \left\{
 \begin{array}{l}
 [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta_r, \\
 [B_1, B_2] + IJ = \zeta_c.
 \end{array}
 \right\}
 }{
 U(k) \text{ gauge invariance}
 }$$

- Without loss of generality, we can set $\zeta_c = 0$, $\zeta_r \neq 0$.

Noncommutative $U(1)$, k -instanton case

- Let us concentrate on the eq. $[B_1, B_2] + IJ = 0$
- One can show $J = 0$.
- $\mathcal{I} = \{P(z, w) \in \mathbb{C}[z, w] \mid P(B_1, B_2) = 0\}$ establishes 1-1 correspondence

codimension k **ideal** of $\mathbb{C}[z, w] \Leftrightarrow U(1)$ NC k -instanton

- rotationally inv. ideal $\mathcal{I}$ are generated by monomials $z^a w^b$.
 $\leadsto \{(a, b) \mid z^a w^b \notin \mathcal{I}\}$ makes a **Young tableau** with k boxes

Example.

z^0w^0	z^0w^1	z^0w^2
z^1w^0	z^1w^1	z^1w^2
z^2w^0		
z^3w^0		

$$\mathbb{C}[z, w]/\mathcal{I} = \text{span}\{1, w, w^2, z, zw, zw^2, z^2, z^3\}$$

$U(N)$ k -instanton case.

- nonzero adjoint VEV $a_i \neq 0 \Leftrightarrow$ D7's are separated along 8,9



- each of the D3s is absorbed on one of D7s



- $U(1)$ k_i -instanton for i -th D7 with $\sum k_i = k$



- N -tuples of Young tableaux with total k boxes

Summary

The instanton contribution to the graviphoton-corrected prepotential is given by

$$\begin{aligned} e^{-\hbar^{-2}\mathcal{F}(a_i)} &= \sum_k e^{-\beta\tau k} (\text{Witten index for } k\text{-instanton moduli}) \\ &= \sum_k e^{-\beta\tau k} \sum_{\substack{(Y_1, \dots, Y_N): \\ N \text{ Young tableaux} \\ \text{with total } k \text{ boxes}}} \prod \dots (\text{complicated exp.}) \end{aligned}$$

Nekrasov's formula

- We have to determine the eigenvalues of $\hbar J + a_i J_i$ on the tangent space.
- They can be calculated by some hard work... One gets finally,

$$e^{-\hbar^{-2}\mathcal{F}(a_i)} = \sum_k e^{-\beta\tau k} \sum_{\substack{(Y_1, \dots, Y_N): \\ N \text{ Young tableaux} \\ \text{with total } k \text{ boxes}}} \sum_{(i,m) \neq (j,n)} \frac{\sinh \frac{\beta}{2}(a_i - a_j + \hbar(y_{i,n} - y_{j,m} + m - n))}{\sinh \frac{\beta}{2}(a_i - a_j + \hbar(m - n))}$$

where $y_{i,n}$ is the length of the i -th row of the N -th Young tableau.

- This corresponds to 5d SYM because finite g_s means finite M-theory circle. Four-dimensional limit is the controlled limit

$$\beta \rightarrow 0, \quad \text{with } \Lambda^{2N} = e^{-\beta\tau} / \beta^{2N} \text{ fixed.}$$

- The formula becomes,

$$e^{-\hbar^{-2}\mathcal{F}(a_i)} = \sum_k \Lambda^{2Nk} \sum_{\substack{(Y_1, \dots, Y_N): \\ N \text{ Young tableaux} \\ \text{with total } k \text{ boxes}}} \sum_{(i,m) \neq (j,n)} \frac{(a_i - a_j + \hbar(y_{i,n} - y_{j,m} + m - n))}{(a_i - a_j + \hbar(m - n))}$$

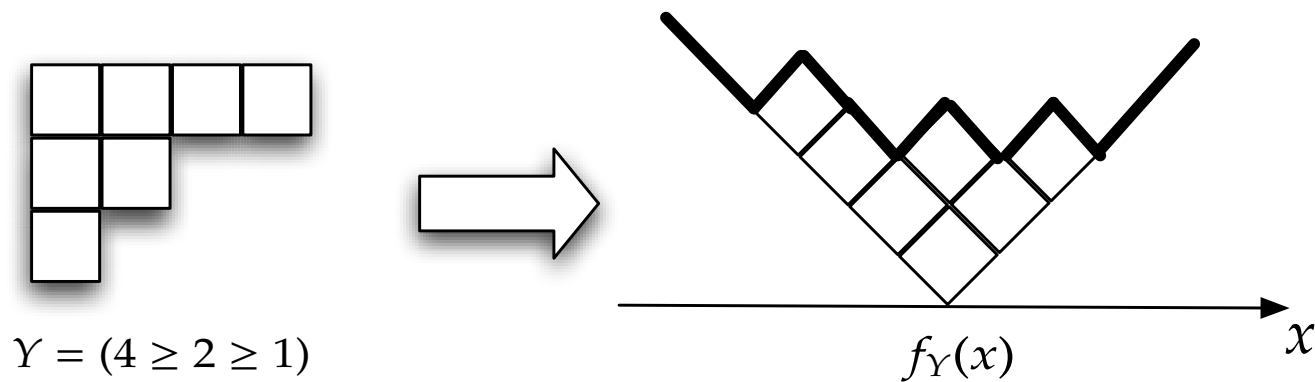
4. Seiberg-Witten curve from the Young Tableaux

Does the Nekrasov's formula agree with good old result by Seiberg and Witten?

We will see that they indeed agree, by transforming the Nekrasov's formula to classical many body systems.

- Define $f_Y(x)$ for a Young tableau by

$$f_Y(x) = |x| + \sum_i \left(|x - \hbar \mathbf{y}_i + \hbar(i-1)| - |x - \hbar \mathbf{y}_i + \hbar i| + |x + \hbar i| - |x + \hbar(i-1)| \right).$$



- Furthermore, define for $\vec{Y} = (Y_1, \dots, Y_N)$ the function $f_{\vec{Y}}$ by

$$f_{\vec{Y}} = \sum_i f_{Y_i}(x - a_i).$$

- Then, Nekrasov's formula can be cast in the form

$$Z_{\text{pert}} Z_{\text{on } \Omega} = \sum_{f_{\vec{Y}}} \exp \left(-\frac{1}{4} \int dx dy f''(x) f''(y) \gamma_{\hbar}(x - y) \right)$$

where

$$\log \frac{x}{\Lambda} = \gamma_{\hbar}(x + \hbar) + \gamma_{\hbar}(x - \hbar) - 2\gamma_{\hbar}(x)$$

$$Z_{\text{pert}} = \exp \left(\sum_{k,l} \gamma_{\hbar}(a_l - a_n) \right).$$

- In the limit $\hbar \rightarrow 0$, it becomes

$$Z_{\text{pert}} Z_{\text{on } \Omega} = \sum_{f_{\vec{Y}}} \exp\left(-\frac{1}{\hbar^2} E[f_{\vec{Y}}]\right)$$

where

$$E[f_{\vec{Y}}] = \frac{1}{8} \int dx dy f''(x) f''(y) (x - y)^2 \left(\log \left(\frac{x - y}{\Lambda} - \frac{3}{2} \right) \right)$$

- Hence, the configuration f_0 which **maximizes** $E[f]$ **dominates** the sum and

$$\mathcal{F}(a_i) = \lim_{\hbar \rightarrow 0} \hbar^2 \log Z = E[f_0].$$

- Note that a_i 's and N are encoded in the asymptotic behavior of $f_{\vec{Y}}$

- It is convenient to Legendre-transform in a_i .

Using the function

$$\xi(x) = \begin{cases} \xi_l x & x \text{ is near } a_l \\ \text{smoothly interpolating} & \text{otherwise} \end{cases}$$

We get $\sum_i \xi_i a_i = \int \xi(x) f''(x) dx$. Hence we have to minimize

$$S[f] = -\frac{1}{8} \int dx dy f''(x) f''(y) (x - y)^2 \left(\log \left(\frac{x - y}{\Lambda} \right) - \frac{3}{2} \right) + \frac{1}{2} \int \xi(x) f''(x) dx.$$

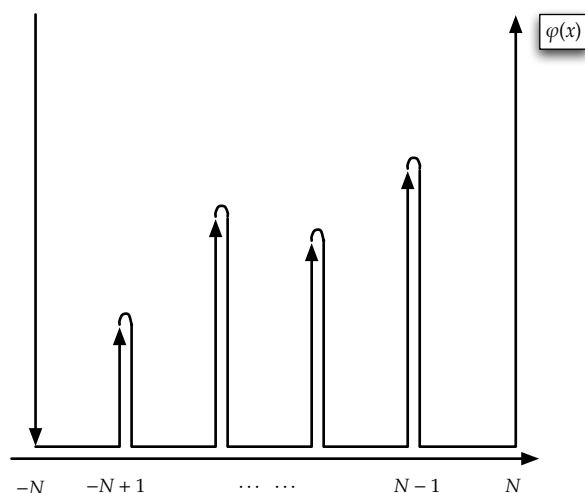
i.e.

$$\underbrace{\int dy (y - x) \left(\log \left(\frac{|x - y|}{\Lambda} \right) - 1 \right) f''(y)}_{\equiv g(x)} = \xi'(x) \quad \dots \clubsuit.$$

- Consider $\phi(x) = f'(x) + g'(x)/(\pi i)$. Differentiating ♣, we get

$$\phi(x)' = \underbrace{\operatorname{Im}\phi(x)'}_{=f''(x)} + i \int \frac{dy}{x-y} \operatorname{Im}\phi(y)'.$$

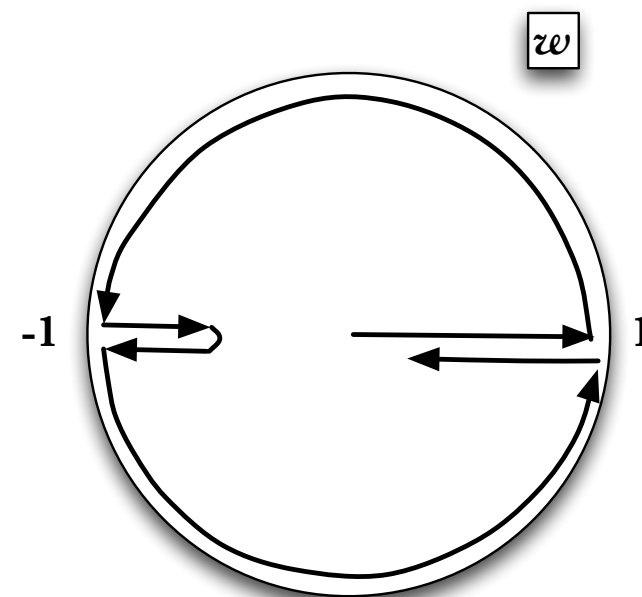
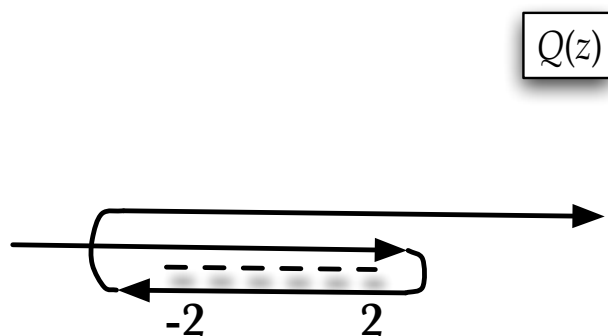
$\leadsto \phi(x)$ can be holomorphically extended to upper half plane. • Boundary behavior of $\phi(x)$ is fixed by ♣:



Hence,

$$\phi(z) = \frac{2}{\pi i} \log w \quad \text{where } w + \frac{1}{w} = \frac{Q(z)}{\Lambda^N}$$

for some N .



- a_i can be recovered from f :

$$a_i = \frac{1}{2} \int_{\text{near } a_i} x f(x)'' dx = \frac{1}{2\pi i} \oint_{A_i} \underbrace{z \frac{dw}{w}}_{\text{SW differential !}}$$

- The SW curve appeared as one solves the maximization problem.

5. Conclusion

- **Prepotential = the (generalized) Witten index of the theory.**
- **The Witten index can be evaluated by the localization.**
- **The fixed points are labeled by Young tableaux.**
- **SW curve appears in the ‘classical limit’ in the sum over Young tableaux.**