Some comments on heterotic string theory

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based on a couple of ongoing projects with various subsets of **Philip Boyle Smith, Justin Kaidi, Ying-Hsuan Lin, Kantaro Ohmori, Vivek Saxena, Mayuko Yamashita, Kazuya Yonekura and Yunqin Zheng.**

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(Words typeset in purple are usually hyperlinked if you download the slides.)

The content of today's talk is not really new (except for the very last part).

Rather, I'd like to **revisit old issues** in string theory
using a more modern point of view,
to understand them better, or at least to shed new lights on them.

So what I'm going to do can be summarized as ...

温	warm
故	the old
而	and then
知	learn
新	the new

[from Analects of Confucius = 論語, 為政第二]

We have all seen this figure: [Polchinski vol.2]

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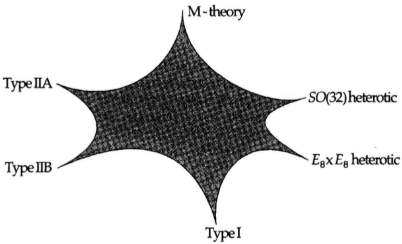

Fig. 14.4. All string theories, and M-theory, as limits of one theory.

Are they really all?

We have all seen this figure: [Polchinski vol.2]

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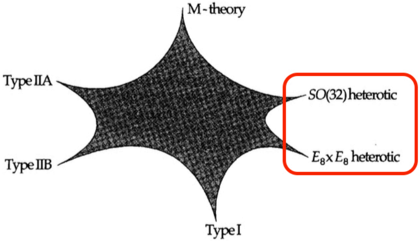


Fig. 14.4. All string theories, and M-theory, as limits of one theory.

Are they really all? Today I'd like to concentrate on these two.

Are there only $SO(32)$ and $E_8 \times E_8$ heterotic strings?

If you are old enough, you might remember reading in textbooks such statement as

One may wonder whether there are other dimension 496 groups that satisfy (13.5.7). The answer is that there are two more. They are $[U(1)]^{496}$ and $E_8 \times [U(1)]^{248}$. These satisfy (13.5.7) rather trivially, since the traces vanish for the $U(1)$ factors. No string theories are known that correspond to either of these groups, and it appears extremely unlikely that any interesting theories can be based on either of them.

from [Green-Schwarz-Witten vol.2, p. 356], first printed in 1987, or

[1] They also hold for $E_8 \times U(1)^{248}$ and $U(1)^{496}$, but no corresponding string theories are known.

from [Polchinski vol.2, p. 101], first printed in 1998.

The argument restricting the gauge groups went as follows.

- Assume 10d $\mathcal{N}=1$ supersymmetry.
- Then the only choice in the spectrum is in the gauge group G.
- Anomaly cancellation via the Green-Schwarz mechanism requires that $\dim G = 496$ and a number of more complicated conditions.
- **Going over all possibilities**†, one only finds

$$E_8 \times E_8, \quad SO(32), \quad U(1)^{496}, \quad U(1)^{248} \times E_8.$$

This was known already to [Green-Schwarz-Witten vol.2, 1987].

† By the way, I didn't know any place where the detail of searching for all possibilities was actually given. So I once assigned a summer undergraduate intern to do just this, whose result is available as [Antonelli 1507.08642]. He is now an astrophysicist, see https://aantonelli94.github.io

More recently, it was found in [Adams-DeWolfe-Taylor 1006.1352] that **the compatibility of**

the requirement of $\mathcal{N}=1$ supersymmetry

and

the structure of the Chern-Simons modification of the B-field

only allows

$$E_8 \times E_8, \quad SO(32)$$

and rules out

$$U(1)^{496}, \quad U(1)^{248} \times E_8.$$

There are also non-supersymmetric heterotic strings,
discussed in [Green-Schwarz-Witten vol.2, Sec.9.5.3]
and in [Polchinski vol.2, Sec.11.3]. If we take the latter, it starts as

11.3 Other ten-dimensional heterotic strings

The other heterotic string theories can all be constructed from a single theory by the twisting construction introduced in section 8.5. The 'least twisted' theory, in the sense of having the smallest number of path integral

A page later, one finds

two, three, four or five of the $\exp(\pi i F_i)$ and forming all products. The first of these produces the $E_8 \times SO(16)$ theory just described; the further twists produce the gauge groups $SO(24) \times SO(8)$, $E_7 \times E_7 \times SO(4)$, $SU(16) \times SO(2)$, and E_8. None of these theories is supersymmetric, and all have tachyons.

After a three-page discussion, one then finds

(NS−, NS+, NS−) sectors. The twists leave an $SO(16) \times SO(16)$ gauge symmetry. Classifying states by their $SO(8)$ spin $\times SO(16) \times SO(16)$ quantum numbers, one finds the massless spectrum

$$\begin{aligned}
(\mathrm{NS}+, \mathrm{NS}+, \mathrm{NS}+): &\quad (\mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{28}, \mathbf{1}, \mathbf{1}) + (\mathbf{35}, \mathbf{1}, \mathbf{1}) \\
&\quad\quad + (\mathbf{8}_v, \mathbf{120}, \mathbf{1}) + (\mathbf{8}_v, \mathbf{1}, \mathbf{120}) , \\
(\mathrm{R}+, \mathrm{NS}-, \mathrm{NS}-): &\quad (\mathbf{8}, \mathbf{16}, \mathbf{16}) , \\
(\mathrm{R}-, \mathrm{R}-, \mathrm{NS}+): &\quad (\mathbf{8}', \mathbf{128}', \mathbf{1}) , \\
(\mathrm{R}-, \mathrm{NS}+, \mathrm{R}-): &\quad (\mathbf{8}', \mathbf{1}, \mathbf{128}') .
\end{aligned}$$

This shows that a tachyon-free theory without supersymmetry is possible.

Construction looks really ad hoc.

Are we sure that there are no others?

For these non-supersymmetric cases,
the spacetime anomaly cancellation doesn't tell us much.

What I'd like to do today is to

classify non-supersymmetric heterotic strings

by applying 2d CFT techniques to the worldsheet.

As a preliminary step, I also need to classify

supersymmetric heterotic strings

using the same worldsheet approach.

Somewhat mysteriously, this classification has some implications on

new exotic non-supersymmetric branes

in supersymmetric heterotic strings ...

Classification of supersymmetric heterotic string theories

What is heterotic string theory?

Bosonic string theory requires **26 dimensions**.

Supersymmetric string theory requires **10 dimensions**.

The worldsheet theory of the heterotic string combines

- **Left-movers** of the bosonic string in **26 dimensions**
- **Right-movers** of the supersymmetric string in **10 dimensions**

$\Longrightarrow$ Needs purely left-moving 2d CFT with $c_L = 26 - 10 = 16$.

(I assume that every CFT I referred to from now on is **unitary**.)

Classification of heterotic string theory
= classification of purely left-moving 2d CFT with $c_L = 16$.

Only two such theories have been known:

$$E_8 \times E_8, \qquad SO(32)$$

(Today I'll be sloppy about the global structure of the groups.)

These two were found via explicit constructions
using free bosons or free fermions.

Have we exhausted all possible free boson
or free fermionic constructions?

Are we sure that there aren't genuinely interacting constructions?

We now have a mathematical proof [Dong-Mason math.QA/0203005] that these two are the only possibilities.

Let me give a physics translation of the proof.

We start from some assumptions.

- We assume that **the theory does not depend on the spin structure** on the 2d worldsheet. More on that later.
- **It should have a partition function**, rather than a partition vector, to be integrated over the moduli space of Riemann surfaces.
- So, random $\mathfrak{g}_k$ won't do: it has multiple conformal blocks on a single Riemann surface.
- **The partition function can still change by a phase factor** under a modular transformation. This comes from the gravitational anomaly.

- The gravitational anomaly polynomial is known to be

$$(c_L - c_R)\frac{p_1}{24}.$$

- Modern theory of anomalies [Freed-Hopkins 1604.06527] says that it needs to be an integer multiple of

$$\frac{p_1}{3}.$$

- We have $c_R = 0$, so $c_L = 8k$.

- Anomaly also dictates the modular transformation law of the torus partition function

$$Z(\tau) = \mathrm{tr}\, q^{L_0 - c_L/24}, \qquad q = e^{2\pi i\tau}$$

to be

$$Z(-\frac{1}{\tau}) = Z(\tau), \qquad Z(\tau+1) = e^{-k(2\pi i/3)} Z(\tau).$$

So you can employ the theory of modular functions.

- We also have $Z(\tau) = q^{-k/3}(1 + O(q))$.
- For $k = 1$ ($c = 8$) and $k = 2$ ($c = 16$) this uniquely fixes the partition function to be :

$$Z(\tau) = \begin{cases} q^{-1/3}(1 + 248q + \cdots), & (c = 8), \\ q^{-2/3}(1 + 496q + \cdots), & (c = 16). \end{cases}$$

Let me repeat:

$$Z(\tau) = \mathrm{tr}\, q^{L_0 - c/24} = \begin{cases} q^{-1/3}(1 + 248q + \cdots), & (c = 8), \\ q^{-2/3}(1 + 496q + \cdots), & (c = 16). \end{cases}$$

So there are 248 or 496 spin-1 operators,
which necessarily form a Lie algebra.

A Lie algebra G of rank r, i.e. containing $U(1)^r$, has $c \geq r$.

Going over all possibilities, one finds that the only solution for $c = 8$ is

$$E_8$$

and the only two solutions for $c = 16$ is

$$E_8 \times E_8, \qquad SO(32).$$

We are done for $G = E_8$ for $c = 8$ or $G = E_8 \times E_8$ for $c = 16$, since

$$Z(\tau) = \chi_{G,\text{vacuum}}.$$

In these cases, and the OPE in the vacuum representation is fixed by the affine Lie algebra symmetry.

For $G = SO(32)$, we have

$$Z(\tau) = \chi_{G,\text{vacuum}} + \chi_{G,\text{spinor}}.$$

There is a unique consistent way to define OPEs in this case, too.
Let us write the resulting theory as $\overline{SO(32)}$, to emphasize that it is not simply the vacuum rep. of the $SO(32)$ affine algebra.

We are done.

Classification of non-supersymmetric heterotic string theories

How do we get spacetime-non-supersymmetric heterotic strings?

People originally did the following: in the lightcone gauge, we have

$$\psi_L^{1,\ldots,32}, \quad X_L^{1,\ldots,8}, \quad X_R^{1,\ldots,8}, \quad \psi_R^{1,\ldots,8}.$$

Now, group the fermions $\psi_L^{1,\ldots,32}$ and $\psi_R^{1,\ldots,8}$ into various subsets, and perform GSO projections for each subsets.

Then you get non-supersymmetric heterotic strings,
when the projections imposed are consistent.

Hard to convince yourself if all possibilities are exhausted!

To be more systematic, recall that the heterotic worldsheet has $\mathcal{N}=(0,1)$ supersymmetry, with a right-moving supercharge Q_R.

The summation over the spin structure for Q_R is part of the string perturbation theory, and **automatically does one GSO projection for the common spin structure of $\psi_R^{1,\ldots,8}$**.

All other GSO projections simply modify the left-moving $c_L = 16$ theory.

So we still have

$$\text{some } c_L = 16 \text{ theory } T, \quad X_L^{1,\ldots,8}, \quad X_R^{1,\ldots,8}, \quad \psi_R^{1,\ldots,8}$$

and we do a single GSO for the common spin structure of $\psi_R^{1,\ldots,8}$.

The only new point compared to the spacetime supersymmetric case is that **the $c_L = 16$ theory T can have a discrete dependence on the spin structure!**

$$\begin{aligned} \text{NS-sector states of } T &\implies \text{spacetime bosons} \\ \text{R-sector states of } T &\implies \text{spacetime fermions} \end{aligned}$$

If T does not depend on the spin structure, i.e. if it is bosonic, the spacetime spectrum becomes **supersymmetric**.

If T does depend on the spin structure, i.e. if it is fermionic, the spacetime spectrum becomes **non-supersymmetric**.

So we now want to classify

purely left-moving **fermionic** CFT with $\boldsymbol{c_L = 16}$.

[Boyle Smith-Lin-YT-Zheng, to appear]	up to $\boldsymbol{c \leq 16}$
[Rayhaun, to appear]	up to $\boldsymbol{c \leq 23}$
[Höhn-Möller, to appear]	up to $\boldsymbol{c = 24}$ & math. rigorous

How do we classify purely left-moving **fermionic** CFT with $c_L = 16$?

We use the **modern theory of bosonization / fermionization**, which says that there is a one-to-one correspondence between

a fermionic theory T_F

and

a bosonic theory T_B with non-anomalous $\mathbb{Z}_2 = \{1, g\}$.

The two theories are related by

$$T_F = \frac{T_B \times \text{Arf}}{g(-1)^{F_{\text{Arf}}}}$$

where the Arf theory (also known as the Kitaev theory) is a theory with one-dimensional Hilbert space such that

$$\begin{aligned} \text{unique NS sector state has} \quad & (-1)^{F_{\text{Arf}}} = +1, \\ \text{unique R sector state has} \quad & (-1)^{F_{\text{Arf}}} = -1. \end{aligned}$$

Conversely

$$T_B = \frac{T_F}{(-1)^F}$$

where orbifolding w.r.t. $(-1)^F$ is simply the sum over spin structure.

[Tachikawa 2018] [Karch-Tong-Turner 1902.05550]

More explicitly,

T_B	untwisted	twisted
$\mathbb{Z}_2$ even	S	U
$\mathbb{Z}_2$ odd	T	V

$$\Updownarrow$$

T_F	NS sector	R sector
$(-1)^F = +1$	S	U
$(-1)^F = -1$	V	T

Basic example is

$$\begin{aligned} T_B &= \text{the Ising model } \sigma \\ T_F &= \text{a Majorana fermion } \psi \end{aligned}$$

So, in some sense,

modern theory of fermionization/bosonization

$=$**generalized Jordan-Wigner transformation.**

Anyhow,

classification of chiral **fermionic** CFT with $c_L = 16$

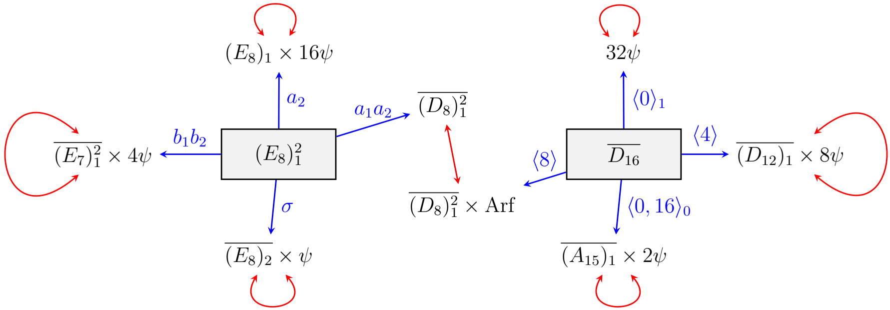
$=$classification of chiral **bosonic** CFT with $c_L = 16$ with $\mathbb{Z}_2$ action.

We saw that only such bosonic theories are

either $E_8 \times E_8$ or $\overline{SO(32)}$.

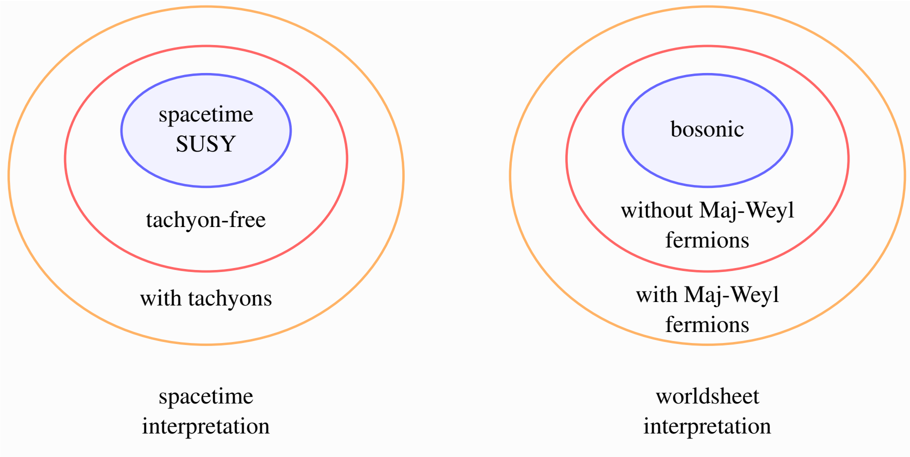
Finding all possible non-anomalous $\mathbb{Z}_2$ actions is
a (somewhat tedious) group theory problem.

The result:



where **blue lines are fermionization/bosonization**
and **red lines are stacking with Arf=Kitaev**.

String theorists in the 80s didn't miss any!


spacetime
SUSY
tachyon-free
with tachyons
spacetime
interpretation
bosonic
without Maj-Weyl
fermions
with Maj-Weyl
fermions
worldsheet
interpretation

T-duality between non-susy and susy heterotic strings

We saw a number of non-supersymmetric heterotic strings.
Where do they belong in this famous diagram?

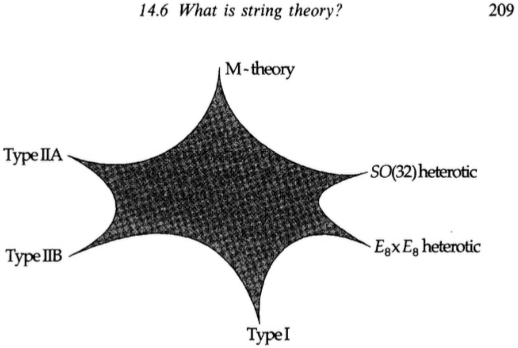


Fig. 14.4. All string theories, and M-theory, as limits of one theory.

In fact they are part of it.

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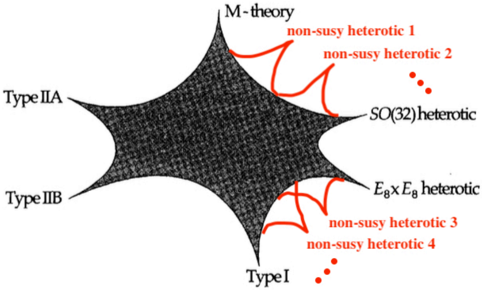


Fig. 14.4. All string theories, and M-theory, as limits of one theory.

because they are T-dual to each other.

[Itoyama-Taylor '87] [Ginsparg-Vafa '87]
[Itoyama-Nakajima 1905.10745] and various papers of [Itoyama, Koga, Nakajima]
[Saxena-YT-Yonekura, to appear?]

Take an example.

$$T_B = E_8 \times E_8 \quad \text{with } \mathbb{Z}_2 \text{ exchanging two factors}$$
$$T_F = (E_8)_2 \times \psi$$

These two were related by fermionization/bosonization:

$$(E_8)_2 \times \psi = \frac{(E_8 \times E_8) \times \text{Arf}}{\text{diag}(\text{exchange}, (-1)^{F_{\text{Arf}}})}$$

and conversely

$$E_8 \times E_8 = \frac{(E_8)_2 \times \psi}{(-1)^F}$$

Correspondingly, we consider

$\boldsymbol{E_8 \times E_8}$ on $\boldsymbol{S^1}$ with **antiperiodic spacetime fermion**
and **the exchange of two $\boldsymbol{E_8}$ factors** around $\boldsymbol{S^1}$

and

$\boldsymbol{(E_8)_2 \times \psi}$ on $\boldsymbol{\tilde{S}^1}$ with **antiperiodic tachyon**.

They are T-dual!

(This is a non-supersymmetric version of CHL strings,
explored recently in [Nakajima 2303.04489].)

How do we see that? Two methods:

- **Work out the spectra on both sides and compare them.**
  outline was given in [Ginsparg-Vafa '87], and details were provided very recently in [Nakajima 2303.04489]

- **More conceptual explanation.** $\Longleftarrow$ **today**

How do we construct the worldsheet theory for

$E_8 \times E_8$ on S^1 with **antiperiodic spacetime fermion**
and **the exchange of two E_8 factors** around S^1 ?

We start from $E_8 \times E_8$ on S^1 with radius $\mathbf{2}R$, and orbifold w.r.t.

$$\text{half-shift} \times \underbrace{(-1)^{F_{\text{spacetime}}}}_{\textbf{how do we do this?}} \times \text{exchange of two } E_8$$

Recall that

$$\begin{aligned} \text{spacetime boson} &= \text{NS sector} \\ \text{spacetime fermion} &= \text{R sector} \end{aligned}$$

so $(-1)^{F_{\text{spacetime}}}$ is **a $\mathbb{Z}_2$ symmetry for the R-sector-ness.**

This is exactly what $(-1)^F$ of the Arf theory does!

Recall the Arf theory has a one-dimensional Hilbert space such that

$$\text{NS sector:} \quad (-1)^F = +1$$
$$\text{R sector:} \quad (-1)^F = -1.$$

So

orbifolding by $(-1)^{F_{\text{spacetime}}}$ =

Adding the Arf theory on the worldsheet

and orbifolding by $(-1)^{F_{\text{Arf}}}$

So

$E_8 \times E_8$ on S^1 of radius R with antiperiodic spacetime fermion and the exchange of two factors around it

equals

$$\frac{S^1_{2R} \times (E_8 \times E_8) \times \mathrm{Arf}}{\mathrm{diag}(\text{half-shift}, \text{exchange}, (-1)^{F_{\mathrm{Arf}}})}.$$

Let us now recall some abstract nonsense:

Suppose a 2d theory A has a $\mathbb{Z}_2$ symmetry.

Then the orbifold $\tilde{A} := A/\mathbb{Z}_2$ has a $\tilde{\mathbb{Z}}_2$ symmetry such that

$$\tilde{A}/\tilde{\mathbb{Z}}_2 = A/\mathbb{Z}_2/\tilde{\mathbb{Z}}_2 = A.$$

For example, let $A = S^1_{2R}$ and $\mathbb{Z}_2$ to be a half-shift. Clearly,

$$A/\mathbb{Z}_2 = S^1_R = \tilde{S}^1_{1/R} = \tilde{A}.$$

By doing the half-shift of the T-dual circle, we have

$$\tilde{A}/\tilde{\mathbb{Z}}_2 = \tilde{S}^1_{1/R}/\tilde{\mathbb{Z}}_2 = \tilde{S}^1_{1/(2R)} = S^1_{2R}.$$

Now, take two theories $\boldsymbol{A}$ and $\boldsymbol{B}$ with $\mathbb{Z}_2$ symmetry.
Then we tautologically have

$$\frac{A \times B}{\text{diagonal } \mathbb{Z}_2} = \frac{\tilde{A} \times \tilde{B}}{\text{diagonal } \tilde{\mathbb{Z}}_2}.$$

We apply it to

$$\frac{\overbrace{S^1_{2R}}^{A} \times \overbrace{(E_8 \times E_8) \times \text{Arf}}^{B}}{\text{diag}(\underbrace{\text{half-shift}}_{\mathbb{Z}_2^A}, \underbrace{\text{exchange}, (-1)^{F_{\text{Arf}}}}_{\mathbb{Z}_2^B})} = \frac{\overbrace{\tilde{S}^1_{1/R}}^{\tilde{A}} \times \overbrace{(E_8)_2 \times \psi}^{\tilde{B}}}{\text{diag}(\underbrace{\text{half-shift}}_{\tilde{\mathbb{Z}}_2^A}, \underbrace{(-1)^F}_{\tilde{\mathbb{Z}}_2^B})}$$

The RHS is the tachyonic $(E_8)_2 \times \psi$ theory with a nontrivial $\mathbb{Z}_2$ action on the tachyon (whose vertex operator is ψ) around S^1.

So we successfully derived

$E_8 \times E_8$ on S^1 with **antiperiodic spacetime fermion**
and **the exchange of two E_8 factors** around S^1

and

$(E_8)_2 \times \psi$ on $\tilde{S}^1$ with **antiperiodic tachyon**.

The same works with all other non-susy heterotic strings.

So we indeed have

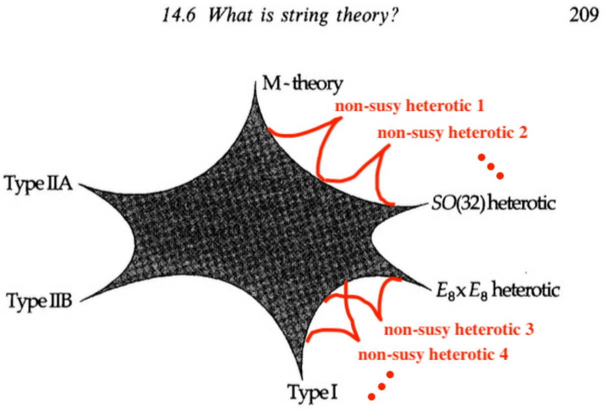


Fig. 14.4. All string theories, and M-theory, as limits of one theory.

Some exotic heterotic branes

We've seen that

non-susy heterotic strings are T-dual to susy heterotic strings.

We can also

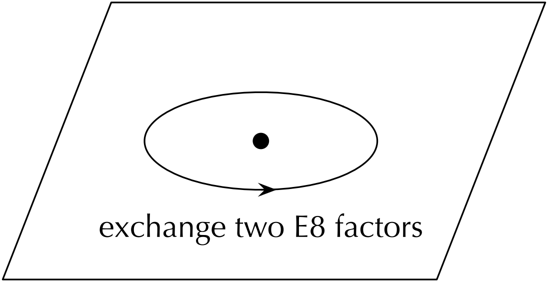
use non-susy heterotic strings

to describe

some non-susy branes in susy heterotic strings.

[Kaidi-Ohmori-YT-Yonekura, to appear]

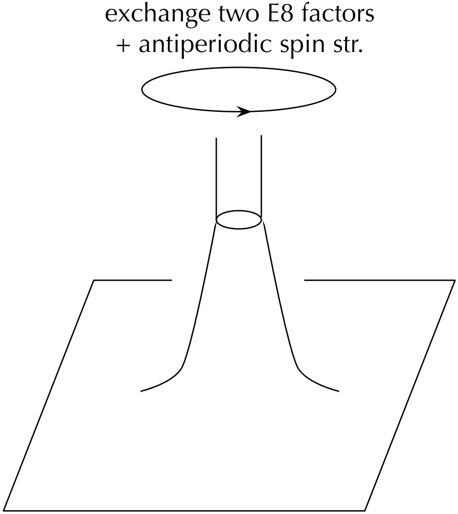
For example, consider the following setup in the $E_8 \times E_8$ heterotic string



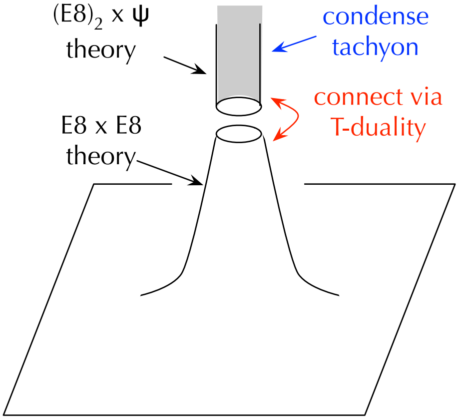
This should give a 7-brane in $E_8 \times E_8$ theory.

How do we analyze it?
We are not very sure, but the following is suggestive:

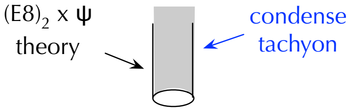
Let's deform like this :



We can then do this



So the *throat* or *core* region is simply



which is

$$\mathbb{R}^{1,7} \times \mathbb{R}_{>0} \times \underbrace{S^1 \times \psi}_{\text{+tachyon deformation}} \times (E_8)_2$$

which would flows in the IR to

$$\mathbb{R}^{1,7} \times \mathbb{R}_{>0}^{\text{linear dilaton}} \times (E_8)_2$$

which is perturbatively stable.

[Hellerman-Swanson 0710.1628] [Kaidi, 2010.10521]

Note that the solution

$$\mathbb{R}^{1,7} \times \mathbb{R}_{>0}^{\text{linear dilaton}} \times (E_8)_2$$

is not too different from the long-known worldsheet description of heterotic NS5-brane, which is

$$\mathbb{R}^{1,5} \times \mathbb{R}_{>0}^{\text{linear dilaton}} \times SU(2)_{-2} \times (E_8)_1 \times (E_8)_1.$$

We can construct other non-supersymmetric heterotic branes in a similar manner:

	in which susy theory?	using which non-susy theory?
7-brane	$E_8 \times E_8$	$\overline{(E_8)_2} \times \psi$
6-brane	$\overline{SO(32)}$	$\overline{SU(16)_1} \times 2\psi$
4-brane	$E_8 \times E_8$	$\overline{(E_7)_1 \times (E_7)_1} \times 4\psi$
0-brane	$\overline{SO(32)}$	$\overline{SO(24)_1} \times 8\psi$

Summary

We classified susy and non-susy heterotic strings in 10d.

All are in the duality web:

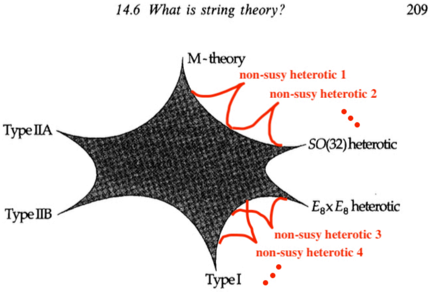


Fig. 14.4. All string theories, and M-theory, as limits of one theory.

Non-susy heterotic strings can be used to construct
a few exotic non-susy branes in susy heterotic strings.

	in which susy theory?	using which non-susy theory?
7-brane	$E_8 \times E_8$	$\overline{(E_8)_2} \times \psi$
6-brane	$\overline{SO(32)}$	$\overline{SU(16)_1} \times 2\psi$
4-brane	$E_8 \times E_8$	$\overline{(E_7)_1 \times (E_7)_1} \times 4\psi$
0-brane	$\overline{SO(32)}$	$\overline{SO(24)_1} \times 8\psi$

Any more questions?