Quantum field theory, mathematics, and their recent interactions

Yuji Tachikawa (Kavli IPMU, U. Tokyo)

KAIST physics colloquium

Apr. 12, 2021

Life between mathematics and theoretical physics

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I am usually categorized as a string theorist.

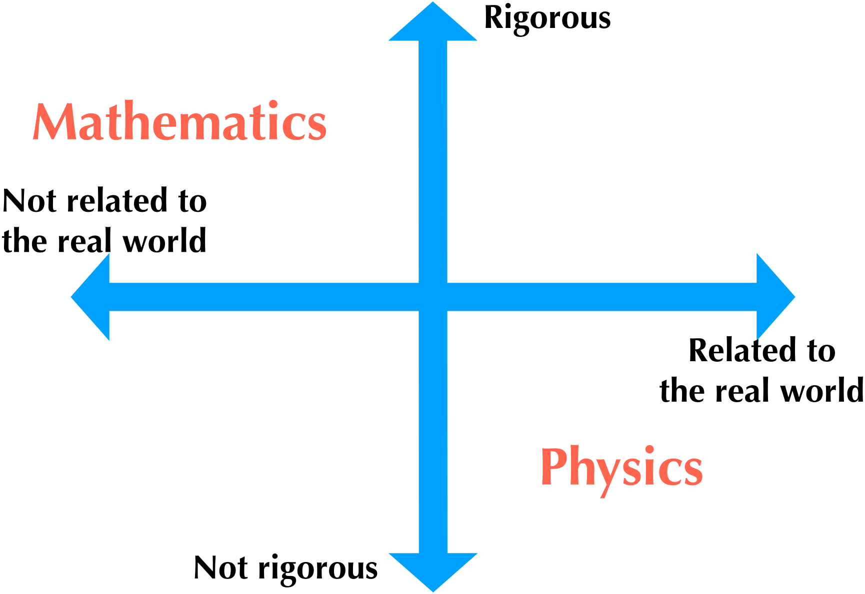
Is what I do **physics**, or **mathematics**, or **mathematical physics**?

I only give a light-hearted analysis of this issue.
For more incisive discussions,
see essays such as Jaffe-Quinn: *Theoretical mathematics*

`https://arxiv.org/abs/math/9307227`

and Thurston: *On proof and progress in mathematics*

`https://arxiv.org/abs/math/9404236`



Rigorous

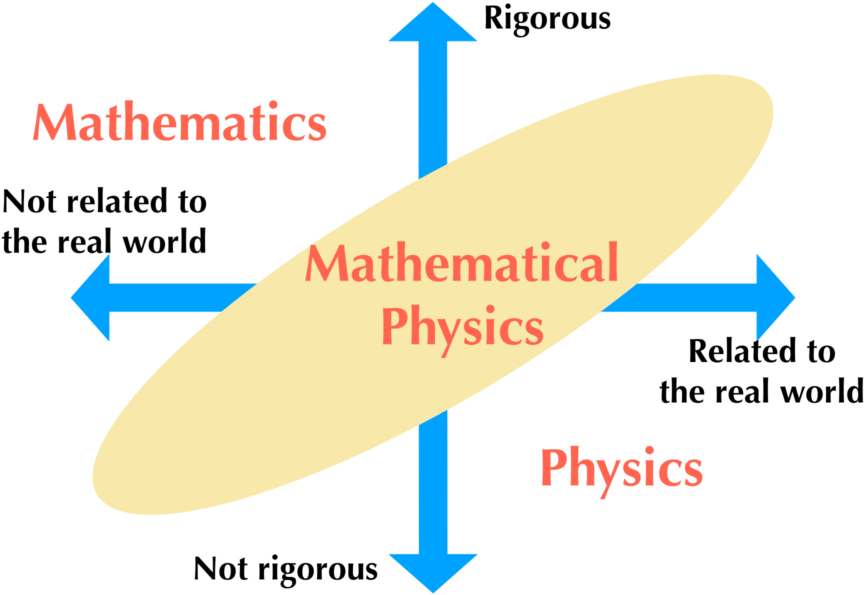
Mathematics

Not related to
the real world

Related to
the real world

Physics

Not rigorous



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**Mathematical
Physics**

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Physics

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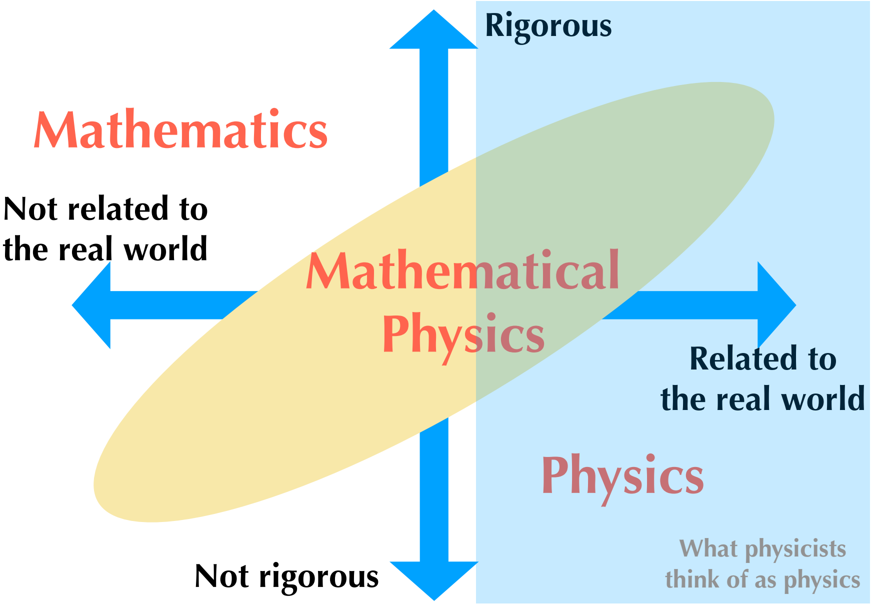

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What physicists
think of as physics

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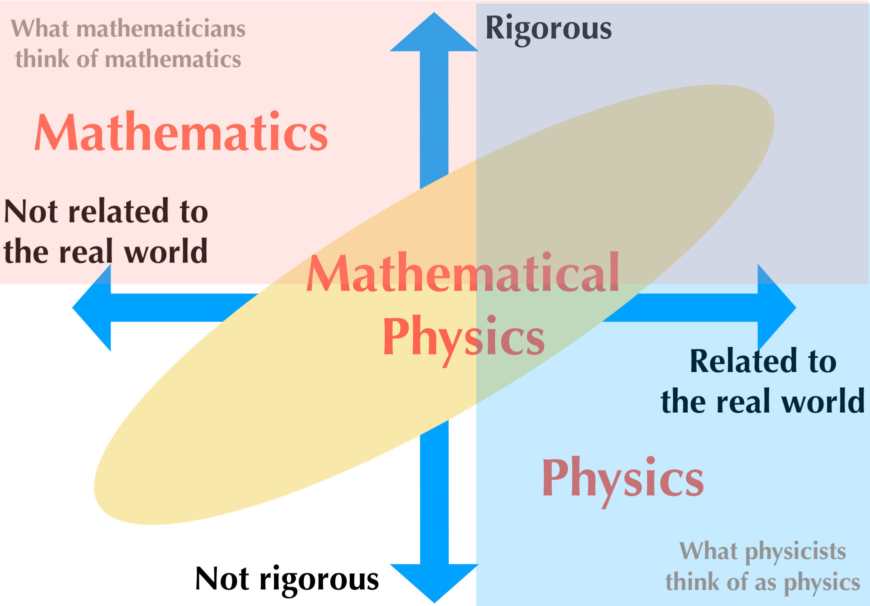


What mathematicians
think of mathematics

Mathematics

Rigorous

**Not related to
the real world**

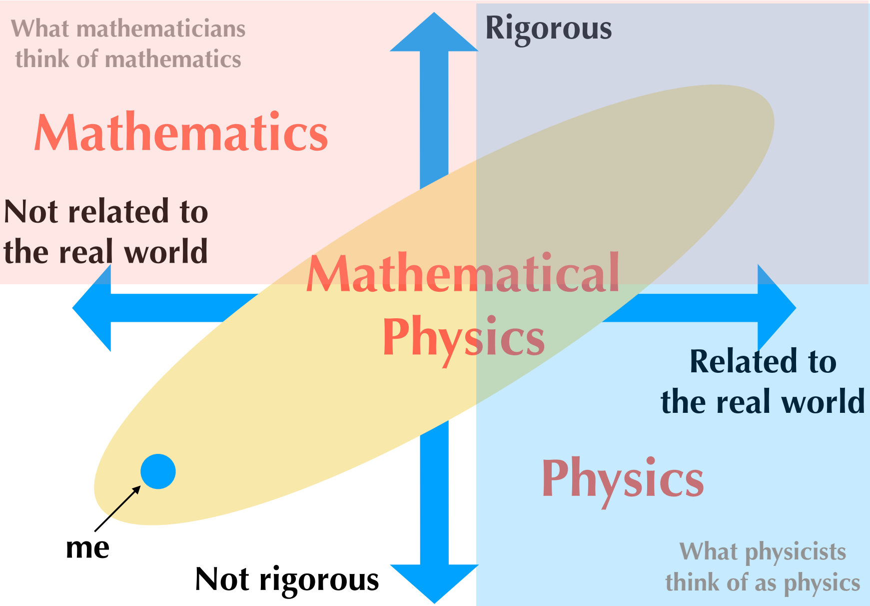
**Mathematical
Physics**

**Related to
the real world**

Physics

Not rigorous

What physicists
think of as physics



What mathematicians think of mathematics

Mathematics

Rigorous

Not related to the real world

Mathematical Physics

Related to the real world

Physics

me

Not rigorous

What physicists think of as physics

Is there any use in something which is **not rigorous**

and **not related to the real world** at the same time?

I want to say **yes**...

Well, I got invited to give a colloquium here by doing that!

Theoretical Physics

Experimental Physics

Work hard

Theoretical Physics

Experimental Physics

Show new experimental results

Theoretical Physics

Experimental Physics

Think about the reason

Theoretical Physics → **Experimental Physics**

Predict new phenomena

Theoretical Physics

Experimental Physics

Work hard

Theoretical Physics

Rigorous Mathematics

Work hard to prove something

Theoretical Physics

Rigorous Mathematics

Present new theorems

Theoretical Physics

Think of their physical significance

Rigorous Mathematics

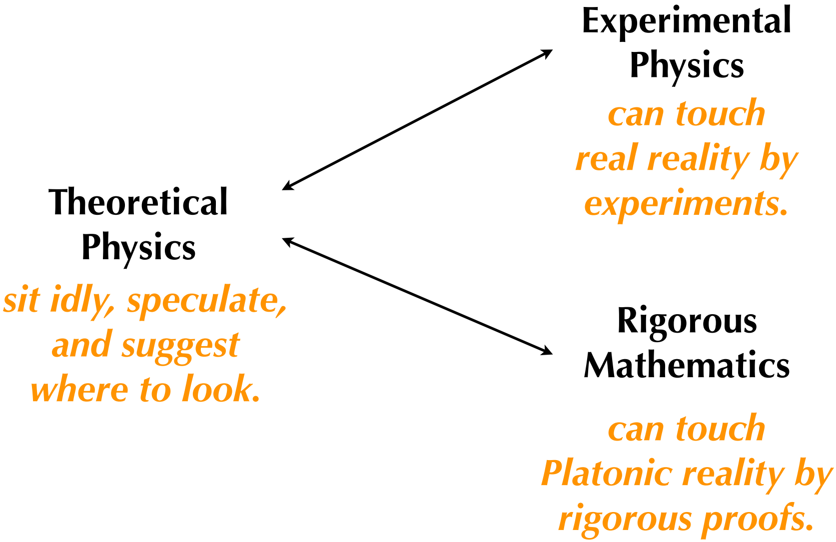
Theoretical Physics $\longrightarrow$ **Rigorous Mathematics**

Predict new theorems

Theoretical Physics

Rigorous Mathematics

Work hard to prove something



Theoretical Physics

sit idly, speculate, and suggest where to look.

Experimental Physics

can touch real reality by experiments.

Rigorous Mathematics

can touch Platonic reality by rigorous proofs.

Does it actually work?

In some cases, yes!

Once upon a time (meaning that it was in the 1980s),
mathematicians were **counting the numbers of spheres**
in a particular **six-dimensional space**, called the **quintic Calabi-Yau**.

size	number	person	
1	2875	J. Harris	(1979)
2	609250	S. Katz	(1986)

It was a laborious process.

I don't know why they were interested in this question!

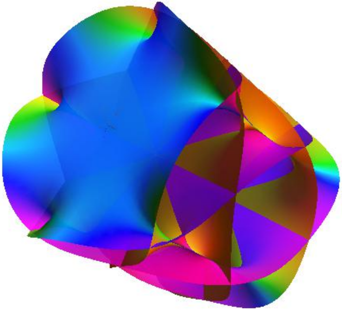
Around the same time, superstring theory was born.

It says the world is $\mathbf{9+1}$ dimensional.

To match with the fact that our world looks $\mathbf{3+1}$ dimensional,
we need to curl up the unwanted $\mathbf{9-3=6}$ dimensions
into a very, very small space.

It was found that the same **Calabi-Yau space** is a nice choice
for this purpose.

So physicists started studying them too.



Slices of the six-dimensional quintic Calabi-Yau
you often see in books and TVs.

`https://www.wolframcloud.com/obj/yuji.tachikawa/Published/calabi_yau_based_on_jakes_code3.nb`

String theorists **P. Candelas, X. de la Ossa, P. Green and L. Parkes** found **in early 1990** a vastly **quicker but non-rigorous method** to compute the number of spheres in the same quintic studied by mathematicians.

Recall:

size	number	person	
$n_1 =$	2875	J. Harris	(1979)
$n_2 =$	609250	S. Katz	(1986)
$n_3 =$	???		

The method of four string theorists involved
differential equations and expanding the solutions in a Taylor series, as physicists would naturally do. They predicted

$$\boldsymbol{n_3 = 317,206,375.}$$

It fell to mathematicians **G. Ellingsrud** and **S. A. Strømme** to test it.
In **June 1990**, they got

$$\boldsymbol{n_3 = 2,682,549,425.}$$

There was a joint math-physics workshop in May 1991
to resolve the issue, so that each side can learn the other side.

There was some progress but the issue remained...

Finally in **July 1991**, Ellingsrud and Strømme found a bug in their computation, and reproduced the prediction by physicists.

$$\boldsymbol{n_3^{\text{phys}} = 317,206,375.}$$
$$\boldsymbol{n_3^{\text{math}} = 317,206,375.}$$

This was when the mathematical field called
the **mirror symmetry** was born.

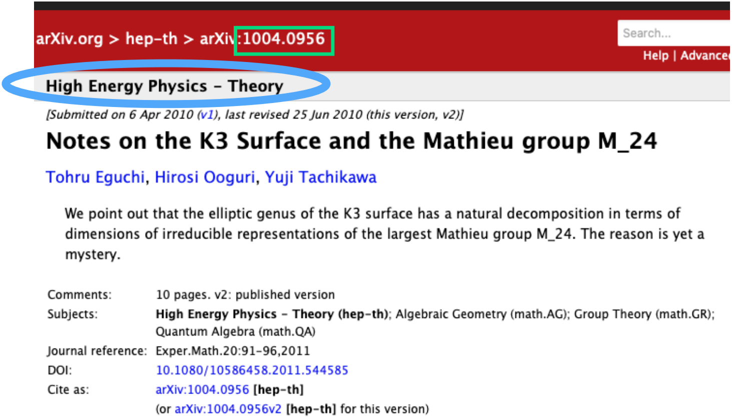
(Details taken from P. Galison, *Mirror symmetry*, in "Growing Explanations," M. Norton Wise ed., Duke University Press, 2004.

https://doi.org/10.1515/9780822390084-002

I thank D. R. Morrison for information.)

There are many other examples of such interactions between mathematics and theoretical physics.

I was lucky to have been involved in one, called the **Mathieu Moonshine**.



High Energy Physics - Theory

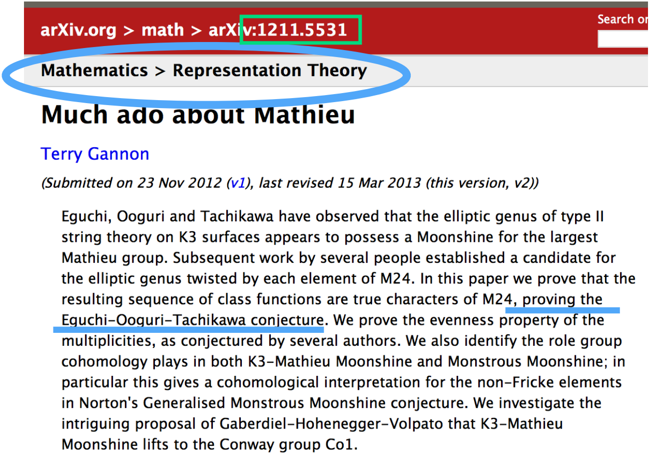
[Submitted on 6 Apr 2010 (v1), last revised 25 Jun 2010 (this version, v2)]

Notes on the K3 Surface and the Mathieu group M_24

Tohru Eguchi, Hirosi Ooguri, Yuji Tachikawa

> We point out that the elliptic genus of the K3 surface has a natural decomposition in terms of dimensions of irreducible representations of the largest Mathieu group M_24. The reason is yet a mystery.

Comments:	10 pages. v2: published version
Subjects:	**High Energy Physics - Theory (hep-th)**; Algebraic Geometry (math.AG); Group Theory (math.GR); Quantum Algebra (math.QA)
Journal reference:	Exper.Math.20:91-96,2011
DOI:	10.1080/10586458.2011.544585
Cite as:	arXiv:1004.0956 **[hep-th]** (or arXiv:1004.0956v2 **[hep-th]** for this version)



arXiv.org > math > arXiv:1211.5531

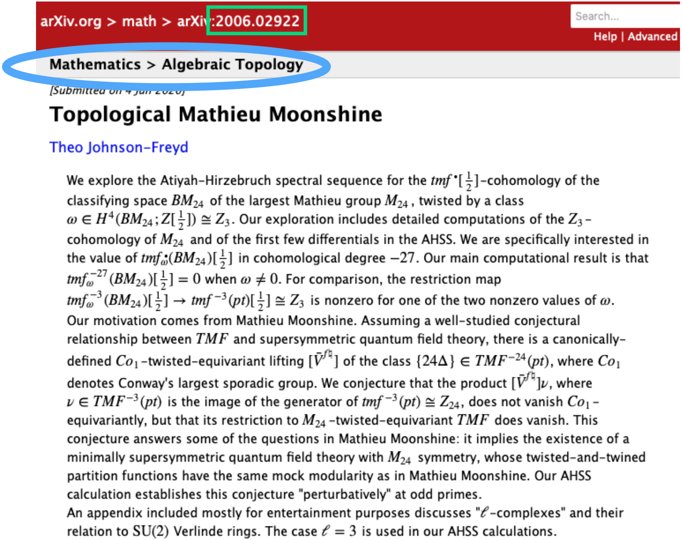
Mathematics > Representation Theory

Much ado about Mathieu

Terry Gannon

(Submitted on 23 Nov 2012 (v1), last revised 15 Mar 2013 (this version, v2))

Eguchi, Ooguri and Tachikawa have observed that the elliptic genus of type II string theory on K3 surfaces appears to possess a Moonshine for the largest Mathieu group. Subsequent work by several people established a candidate for the elliptic genus twisted by each element of M24. In this paper we prove that the resulting sequence of class functions are true characters of M24, proving the Eguchi-Ooguri-Tachikawa conjecture. We prove the evenness property of the multiplicities, as conjectured by several authors. We also identify the role group cohomology plays in both K3-Mathieu Moonshine and Monstrous Moonshine; in particular this gives a cohomological interpretation for the non-Fricke elements in Norton's Generalised Monstrous Moonshine conjecture. We investigate the intriguing proposal of Gaberdiel-Hohenegger-Volpato that K3-Mathieu Moonshine lifts to the Conway group Co1.



arXiv.org > math > arXiv:2006.02922

Mathematics > Algebraic Topology

[Submitted on 4 Jun 2020]

Topological Mathieu Moonshine

Theo Johnson-Freyd

We explore the Atiyah-Hirzebruch spectral sequence for the $tmf^\bullet[\frac{1}{2}]$-cohomology of the classifying space BM_{24} of the largest Mathieu group M_{24}, twisted by a class $\omega \in H^4(BM_{24};Z[\frac{1}{2}]) \cong Z_3$. Our exploration includes detailed computations of the Z_3-cohomology of M_{24} and of the first few differentials in the AHSS. We are specifically interested in the value of $tmf_\omega^\bullet(BM_{24})[\frac{1}{2}]$ in cohomological degree -27. Our main computational result is that $tmf_\omega^{-27}(BM_{24})[\frac{1}{2}] = 0$ when $\omega \neq 0$. For comparison, the restriction map $tmf_\omega^{-3}(BM_{24})[\frac{1}{2}] \to tmf^{-3}(pt)[\frac{1}{2}] \cong Z_3$ is nonzero for one of the two nonzero values of ω.
Our motivation comes from Mathieu Moonshine. Assuming a well-studied conjectural relationship between TMF and supersymmetric quantum field theory, there is a canonically-defined Co_1-twisted-equivariant lifting $[\bar{V}^{f\natural}]$ of the class $\{24\Delta\} \in TMF^{-24}(pt)$, where Co_1 denotes Conway's largest sporadic group. We conjecture that the product $[\bar{V}^{f\natural}]\nu$, where $\nu \in TMF^{-3}(pt)$ is the image of the generator of $tmf^{-3}(pt) \cong Z_{24}$, does not vanish Co_1-equivariantly, but that its restriction to M_{24}-twisted-equivariant TMF does vanish. This conjecture answers some of the questions in Mathieu Moonshine: it implies the existence of a minimally supersymmetric quantum field theory with M_{24} symmetry, whose twisted-and-twined partition functions have the same mock modularity as in Mathieu Moonshine. Our AHSS calculation establishes this conjecture "perturbatively" at odd primes.
An appendix included mostly for entertainment purposes discusses "ℓ-complexes" and their relation to SU(2) Verlinde rings. The case $\ell = 3$ is used in our AHSS calculations.

I would like to give some detail of this **Mathieu Moonshine**, but it would be a long way to go. I need to tell you

- what is **quantum field theory**,
- what is **string theory**,
- what are the **Mathieu groups**, and
- what is the **moonshine**.

Let me try.

What is **Quantum** **Field** Theory = QFT ?

- Describes **quantum** properties of **fields**, where
- **fields** are **anything which extend along space and time**, such as
- electromagnetic fields (=light), crystal vibration, electron fields …

The prototypical example is the **Quantum Electrodynamics** (QED):

- describes quantized electromagnetic fields interacting with charged particles
- was established around 1950s
- with many developments since then

Theory and experiment match extremely well in QED.

The prime example is the **anomalous magnetic moment of electron**:

$$\begin{aligned} \boldsymbol{a_e^{\text{theory}}} &= \mathbf{0.01\,159\,652\,181\ldots} \\ \boldsymbol{a_e^{\text{experiment}}} &= \mathbf{0.01\,159\,652\,181\ldots} \end{aligned}$$

I can say that:

- QFT is **well researched**,
- QFT predicts quantities with **high precision**, and
- QFT **agrees very well with experiment**.

But it is **mathematically incomplete**, in that
no satisfactory formalization is known.

Quantum mechanics and general relativity are OK for mathematicians,
but QFT is not.

It is analogous to the situation in the past:

Ancient Egyptians could build pyramids,
although they had not formalized geometry.

Physicists can compute things although they have not formalized QFT.

The flip side of the coin is that QFT might produce
new results in mathematics.

Let us move on to **string theory**.

It is a **quantum** theory of **strings moving relativistically**. It turns out that:

- it is consistent only in **9+1 dimensions**
- it automatically contains **quantum gravity**

Reconciling gravity and quantum mechanics is one of the long-standing problems in physics. There are many competing approaches.

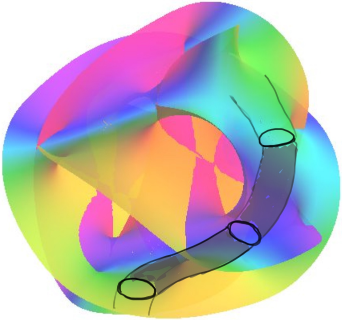
There are many who are passionately for string theory,
and also many who are passionately against string theory.

Does it describe the real world? I do not know.

For me what matters is whether it is Platonically consistent.

It seems it is. And many mathematical predictions have come out of it.

Suppose you want to study strings moving in a Calabi-Yau...



Please excuse my bad drawing.

It is done in terms of 1+1 dimensional QFT on the worldsheet.

Mathieu Moonshine concerns strings moving in **K3 space**.

It is a **closed four-dimensional** space satisfying

$$\frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma} = R_{\mu\nu\rho\sigma}$$

which is not completely flat.

You can think of it as a space which is **half flat**,
in a precise technical sense.

In particular, it solves the vacuum Einstein equation
with **zero cosmological constant**.

K3 is named by André Weil, honoring three mathematicians **Kähler**, **Kummer**, and **Kodaira**, and also after the beautiful mountain **K2**:

https://en.wikipedia.org/wiki/K2

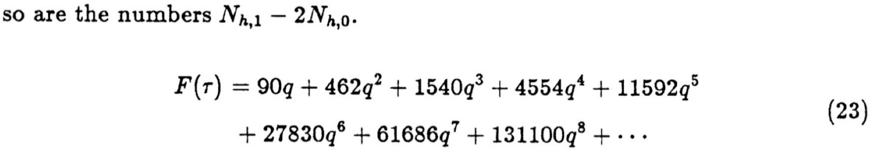
The origin of Mathieu Moonshine goes back to the work by Eguchi and Ooguri in 1989.

T. Eguchi

H. Ooguri

Ooguri was preparing his PhD thesis under Eguchi, studying strings moving in K3.

A central result in his PhD thesis is this:



so are the numbers $N_{h,1} - 2N_{h,0}$.

$$F(\tau) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 \\ + 27830q^6 + 61686q^7 + 131100q^8 + \cdots \tag{23}$$

This is basically the partition function of a string moving in K3.

What does it mean?

https://ooguri.caltech.edu/documents/8002/phd_thesis.pdf

To understand it, we now need to turn to
the role of **symmetry** in **quantum mechanics**.

Everybody will learn / learned that the angular momentum in quantum mechanics takes the value

$$j_z = \underbrace{-j, -j+1, \ldots, j-1, j}_{2j+1 \text{ choices}}.$$

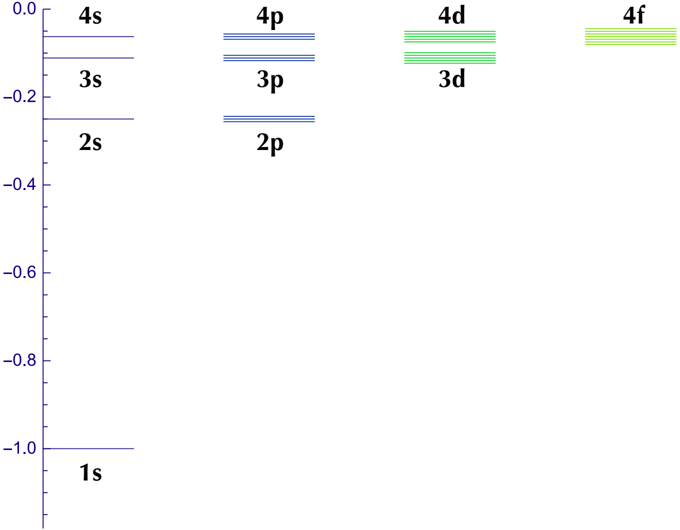
So

$$\begin{array}{rcrl} j=0 & \Rightarrow & j_z = & 0 \\ j=\frac{1}{2} & \Rightarrow & j_z = & -\frac{1}{2}, +\frac{1}{2} \\ j=1 & \Rightarrow & j_z = & -1, 0, +1 \\ j=\frac{3}{2} & \Rightarrow & j_z = & -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2} \end{array}$$

For integer j, there are also traditional names

j	0	1	2	3	$\cdots$
name	s	p	d	f	$\cdots$
degeneracy	1	3	5	7	$\cdots$

The spectrum of a hydrogen atom, to the zeroth approximation, looks like this:



The **degeneracy** is related to the **symmetry**.

The angular momentum operators $L_{x,y,z}$ are
infinitesimal generators of the three dimensional **rotation group** $so(3)$,
the symmetry of the hydrogen atom.

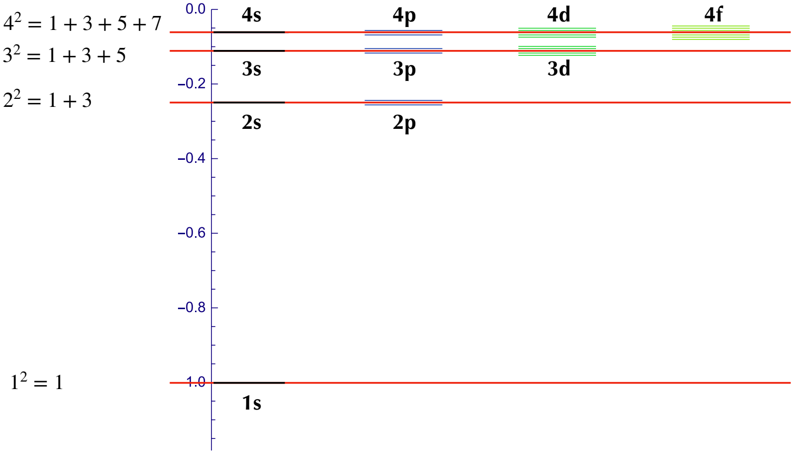
representation=	j	0	1	2	3	$\cdots$
	name	s	p	d	f	$\cdots$
dimension=	degeneracy	1	3	5	7	$\cdots$

The total angular momentum j specifies
how the symmetry acts on the quantum states.

Equivalently, it specifies the **representation** of the symmetry group.

The degeneracy is also known as the **dimension**.

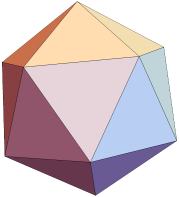
The spectrum of a hydrogen atom, to the zeroth approximation, shows accidental degeneracies **in addition to the rotational symmetry**:



It is known to reflect a hidden 4-dimensional rotational symmetry $\boldsymbol{so}(4)$ [Pauli, Fock, …]

The rotation group is a continuous group.

There are also finite groups, e.g. the symmetry A_5 of



which contains 60 elements.

For the symmetry $\boldsymbol{A_5}$ to act on quantum mechanical systems, it needs to be **represented** by matrices.

$$\boldsymbol{A_5 \ni g \mapsto \rho(g) = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}}$$

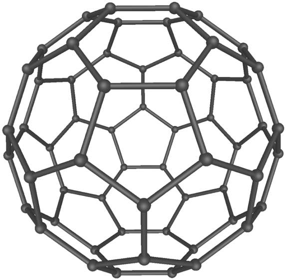
The size is known as the **dimension**, which is three in this example.

Irreducible representations and their dimensions of $\boldsymbol{A_5}$ are known:

name	$\boldsymbol{A}$	$\boldsymbol{T_1}$	$\boldsymbol{T_2}$	$\boldsymbol{G}$	$\boldsymbol{H}$
dimension	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$

(There are two different irreducible representations with the same dimension 3).

We physicists are perfectly happy studying various concrete systems with various concrete symmetries. This is the schematic structure of C_{60}, the **fullerene**, from Wikipedia:



https://en.wikipedia.org/wiki/Fullerene

The properties of A_5 is very useful (and essential) when studying its electronic properties, etc.

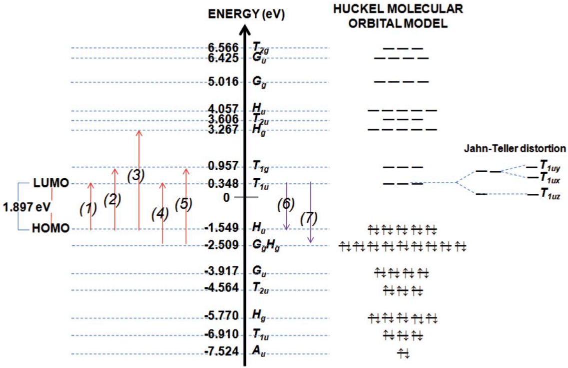


Figure 7: Schematic of the electronic structure of C_{60} as calculated by the Hückel model [80, 81] and its possible electronic excitation transitions experimentally observed in this study. The electronic excitation and emission transitions were numbered in the bracket as *(1)* to *(5)* and *(6)* to *(7)*, respectively.

from T. E. Saraswati et al., *The Study of the Optical Properties of C60 Fullerene in Different Organic Solvents*, Open Chem. 17 (2019) 119–1212

https://doi.org/10.1515/chem-2019-0117

Mathematicians think differently:

Let's classify all possible symmetries, say **all finite groups**.

Any **finite group** is made out of **finite simple groups**,
just as any **integer** is a product of **prime numbers**.

So they say: let us **classify finite simple groups** first.

Classification of finite simple groups:

- **Cyclic** group of prime order $\mathbb{Z}_p$, $p = 2, 3, 5, \dots$
- **Alternating** groups A_5, A_6, …
- Finite groups of **Lie type**,
  obtained by considering continuous groups over finite fields,
- and finally, the 26 **sporadic groups**.

Classification of finite simple groups:

The proof is said to be the **longest in the history** of mathematics. Originally announced to be complete in the late 1970s to early 1980s with papers and preprints said to **total 5000 pages**.

https://doi.org/10.1090/S0273-0979-1979-14551-8

A streamlined rewrite of the entire proof in a single series of volumes is going on for decades. It already has about **3500 pages**,
but is yet not complete.

https://www.ams.org/publications/authors/books/postpub/surv-40

https://www.ams.org/journals/notices/201806/rnoti-p646.pdf

Classification of finite simple groups:

- Cyclic group of prime order $\mathbb{Z}_p$, $p = 2, 3, 5, \ldots$
- Alternating groups A_5, A_6, ...
- **Finite groups of Lie type**
- and finally, the 26 sporadic groups.

The last two series of finite groups of Lie type were found by Rimhak Ree (이임학, 李林學) in 1960/1961.

`https://doi.org/10.1090/S0002-9904-1960-10523-X`

`https://doi.org/10.1090/S0002-9904-1961-10527-2`

`https://dx.doi.org/10.5169/seals-685366`

They are called Ree groups.

Lie groups and Ree groups are difficult to distinguish for Koreans and Japanese alike, since we don't have distinctions between /r/ and /l/...

(Prof. Rimhak Ree, 1922–2005)

http://news.khan.co.kr/kh_news/khan_art_view.html?artid=201510302151325

https://horizon.kias.re.kr/13561/

Classification of finite simple groups:

- Cyclic group of prime order $\mathbb{Z}_p$, $p = 2, 3, 5, \ldots$
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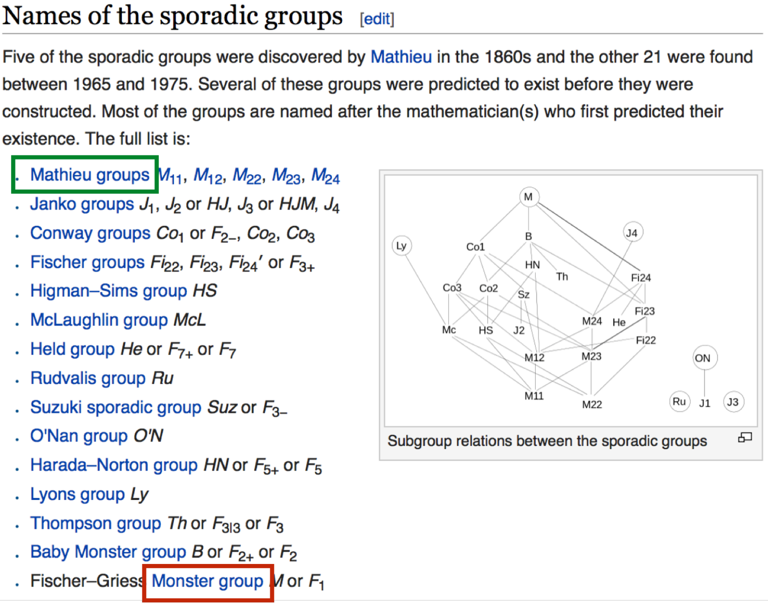


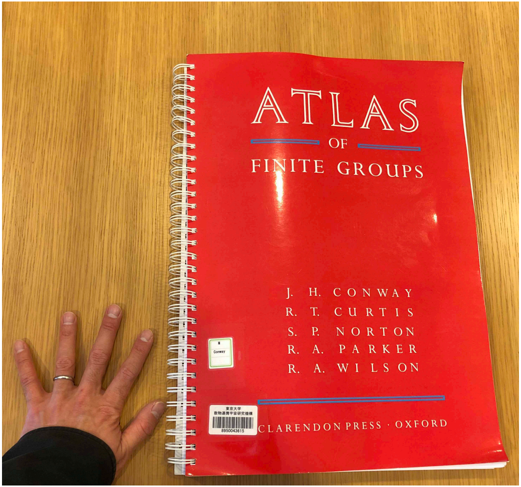
Names of the sporadic groups [edit]

Five of the sporadic groups were discovered by Mathieu in the 1860s and the other 21 were found between 1965 and 1975. Several of these groups were predicted to exist before they were constructed. Most of the groups are named after the mathematician(s) who first predicted their existence. The full list is:

- Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}
- Janko groups J_1, J_2 or HJ, J_3 or HJM, J_4
- Conway groups Co_1 or F_{2-}, Co_2, Co_3
- Fischer groups Fi_{22}, Fi_{23}, Fi_{24}' or F_{3+}
- Higman–Sims group HS
- McLaughlin group McL
- Held group He or F_{7+} or F_7
- Rudvalis group Ru
- Suzuki sporadic group Suz or F_{3-}
- O'Nan group $O'N$
- Harada–Norton group HN or F_{5+} or F_5
- Lyons group Ly
- Thompson group Th or $F_{3|3}$ or F_3
- Baby Monster group B or F_{2+} or F_2
- Fischer–Griess Monster group M or F_1

Subgroup relations between the sporadic groups

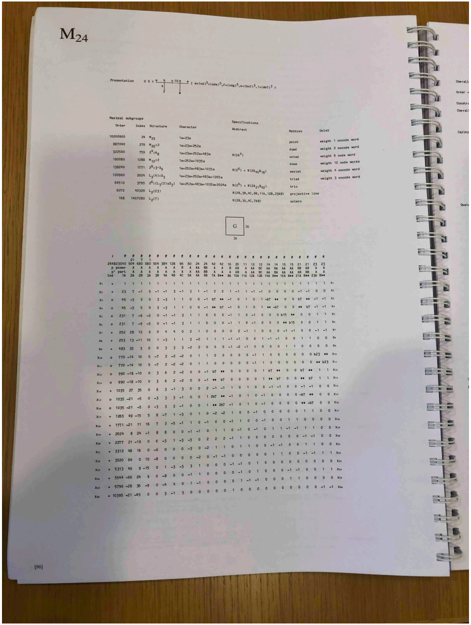
When it comes to the data of finite groups, nothing can beat **the ATLAS**:



ATLAS
OF
FINITE GROUPS

J. H. CONWAY
R. T. CURTIS
S. P. NORTON
R. A. PARKER
R. A. WILSON

CLARENDON PRESS · OXFORD



M_{24}

Presentation $G \cong \langle a - b - c \overset{10}{-} d - e,\ b \overset{4}{-} f,\ d - g \mid a=(cd)^5=(cde)^5, f=(cdg)^5, e=(bcf)^3, 1=(abf)^3 \rangle$

Maximal subgroups

Order	Index	Structure	Character	Specifications Abstract	Mathieu	Golay
10200960	24	M_{23}	1a+23a		point	weight 1 cocode word
887040	276	$M_{22}:2$	1a+23a+252a		duad	weight 2 cocode word
322560	759	$2^4:A_8$	1a+23a+252a+483a	$N(2A^4)$	octad	weight 8 code word
190080	1288	$M_{12}:2$	1a+252a+1035a		duum	weight 12 code words
138240	1771	$2^6:3{\cdot}S_6$	1a+252a+483a+1035a	$N(2^6) = N(2A_{45}B_{18})$	sextet	weight 4 cocode word
120960	2024	$L_3(4):S_3$	1a+23a+252a+483a+1265a		triad	weight 3 cocode word
64512	3795	$2^6:(L_3(2)\times S_3)$	1a+252a+483a+1035a+2024a	$N(2^6) = N(2A_{21}B_{42})$	trio	
6072	40320	$L_2(23)$		N(2B,3B,4C,6B,11A,12B,23AB)	projective line	
168	1457280	$L_2(7)$		N(2B,3A,4C,7AB)	octern	

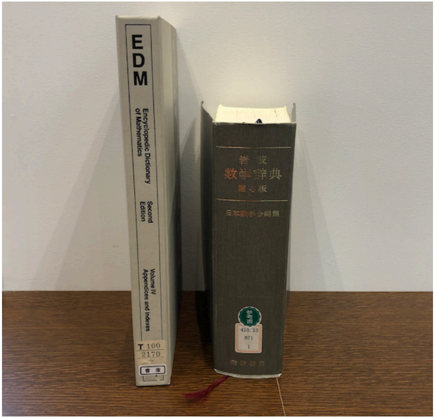
$\boxed{G}$ 26

26

	;	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	
		244823040	21504	7680	1080	504	384	128	96	60	24	24	42	42	16	20	11	12	12	14	14	15	15	21	21	23	23	
	p power		A	A	A	A	A	A	B	A	AA	BB	A	A	B	AB	A	AA	BC	AA	BA	AA	AA	BB	AB	A	A	
	p' part		A	A	A	A	A	A	A	A	AA	BB	A	A	A	AB	A	AA	BC	AA	BA	AA	AA	AB	BB	A	A	
	ind	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	B**	8A	10A	11A	12A	12B	14A	B**	15A	B**	21A	B**	23A	B**	
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	χ_1
χ_2	+	23	7	-1	5	-1	-1	3	-1	3	1	-1	2	2	1	-1	1	-1	-1	0	0	0	0	-1	-1	0	0	χ_2
χ_3	o	45	-3	5	0	3	-3	1	1	0	0	-1	b7	**	-1	0	1	0	1	-b7	**	0	0	b7	**	-1	-1	χ_3
χ_4	o	45	-3	5	0	3	-3	1	1	0	0	-1	**	b7	-1	0	1	0	1	**	-b7	0	0	**	b7	-1	-1	χ_4
χ_5	o	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	b15	**	0	0	1	1	χ_5
χ_6	o	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	**	b15	0	0	1	1	χ_6
χ_7	+	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	-1	1	0	0	0	-1	-1	0	0	-1	-1	χ_7
χ_8	+	253	13	-11	10	1	-3	1	1	3	-2	1	1	1	-1	-1	0	0	1	-1	-1	0	0	1	1	0	0	χ_8
χ_9	+	483	35	3	6	0	3	3	3	-2	2	0	0	0	-1	-2	-1	0	0	0	0	1	1	0	0	0	0	χ_9
χ_{10}	o	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	b23	**	χ_{10}
χ_{11}	o	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	**	b23	χ_{11}
χ_{12}	o	990	-18	-10	0	3	6	2	-2	0	0	-1	b7	**	0	0	0	0	1	b7	**	0	0	b7	**	1	1	χ_{12}
χ_{13}	o	990	-18	-10	0	3	6	2	-2	0	0	-1	**	b7	0	0	0	0	1	**	b7	0	0	**	b7	1	1	χ_{13}
χ_{14}	+	1035	27	35	0	6	3	-1	3	0	0	2	-1	-1	1	0	1	0	0	-1	-1	0	0	-1	-1	0	0	χ_{14}
χ_{15}	o	1035	-21	-5	0	-3	3	3	-1	0	0	1	2b7	**	-1	0	1	0	-1	0	0	0	0	-b7	**	0	0	χ_{15}
χ_{16}	o	1035	-21	-5	0	-3	3	3	-1	0	0	1	**	2b7	-1	0	1	0	-1	0	0	0	0	**	-b7	0	0	χ_{16}
χ_{17}	+	1265	49	-15	5	8	-7	1	-3	0	1	0	-2	-2	1	0	0	-1	0	0	0	0	0	1	1	0	0	χ_{17}
χ_{18}	+	1771	-21	11	16	7	3	-5	-1	1	0	-1	0	0	-1	1	0	0	-1	0	0	1	1	0	0	0	0	χ_{18}
χ_{19}	+	2024	8	24	-1	8	8	0	0	-1	-1	0	1	1	0	-1	0	-1	0	1	1	-1	-1	1	1	0	0	χ_{19}
χ_{20}	+	2277	21	-19	0	6	-3	1	-3	-3	0	2	2	2	-1	1	0	0	0	0	0	0	0	-1	-1	0	0	χ_{20}
χ_{21}	+	3312	48	16	0	-6	0	0	0	-3	0	-2	1	1	0	1	1	0	0	-1	-1	0	0	1	1	0	0	χ_{21}
χ_{22}	+	3520	64	0	10	-8	0	0	0	0	-2	0	-1	-1	0	0	0	0	0	1	1	0	0	-1	-1	1	1	χ_{22}
χ_{23}	+	5313	49	9	-15	0	1	-3	-3	3	1	0	0	0	-1	-1	0	1	0	0	0	0	0	0	0	0	0	χ_{23}
χ_{24}	+	5544	-56	24	9	0	-8	0	0	-1	1	0	0	0	0	-1	0	1	0	0	0	-1	-1	0	0	1	1	χ_{24}
χ_{25}	+	5796	-28	36	-9	0	-4	4	0	1	-1	0	0	0	0	1	-1	-1	0	0	0	1	1	0	0	0	0	χ_{25}
χ_{26}	+	10395	-21	-45	0	0	3	-1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	-1	-1	0	0	χ_{26}

[96]

Although not as comprehensive as the Atlas,
this Japanese math encyclopedia is also quite useful:



which has tables of representations of *some* finite groups.

Mathieu groups: $\boldsymbol{M_{11}}$, $\boldsymbol{M_{12}}$, $\boldsymbol{M_{22}}$, $\boldsymbol{M_{23}}$ and $\boldsymbol{M_{24}}$

Found by Mathieu in 1861 and 1873.

The largest of them, $\boldsymbol{M_{24}}$, has $\mathbf{244823040}$ elements.

It is the **symmetry** of the extended binary **Golay code**,
introduced in 1949, only one year after
Shannon introduced the information theory.

Notes on Digital Coding*

The consideration of message coding as a means for approaching the theoretical capacity of a communication channel, while reducing the probability of errors, has suggested the interesting number theoretical problem of devising lossless binary (or other) coding schemes serving to insure the reception of a correct, but reduced, message when an upper limit to the number of transmission errors is postulated.

An example of lossless binary coding is treated by Shannon[1] who considers the case of blocks of seven symbols, one or none of which can be in error. The solution of this case can be extended to blocks of 2^n-1-binary symbols, and, more generally, when coding schemes based on the prime number p are employed, to blocks of $p^n-1/p-1$ symbols which are transmitted, and received with complete equivocation of one or no symbol, each block comprising n redundant symbols designed to remove the equivocation. When encoding the message, the n redundant symbols x_m are determined in terms of the message symbols Y_k from the congruent relations

$$E_m \equiv X_m + \sum_{k=1}^{k=(p^n-1)/\,p-1)-n} a_{mk}\,Y_k \equiv 0 \pmod{p}.$$

In the decoding process, the E's are recalculated with the received symbols, and their ensemble forms a number on the base p which determines univocally the mistransmitted symbol and its correction.

In passing from n to $n+1$, the matrix with n rows and $p^n-1/p-1$ columns formed with the coefficients of the X's and Y's in the expression above is repeated p times horizontally, while an $(n+1)$ st row added, consisting of $p^n-1/p-1$ zeroes, followed by as many one's etc. up to $p-1$; an added column of n zeroes with a one for the lowest term completes the new matrix for $n+1$.

If we except the trivial case of blocks of $2S+1$ binary symbols, of which any group comprising up to S symbols can be received in error which equal probability, it does not appear that a search for lossless coding schemes, in which the number of errors is limited but larger than one, can be systematized so as to yield a family of solutions. A necessary but not sufficient condition for the existence of such a lossless coding scheme in the binary system is the existence of three or more first numbers of a line of Pascal's triangle which add up to an exact power of 2. A limited search has revealed two such cases; namely, that of the first three numbers of the 90th line, which add up to 2^{12} and that of the first four numbers of the 23rd line, which add up to 2^{11}. The first case does not correspond to a lossless coding scheme, for, were such a scheme to exist, we could designate by r the number of E_m ensembles corresponding to one error and having an odd number of 1's and by $90-r$ the remaining (even) ensembles. The odd ensembles corresponding to two transmission errors could be formed by re-entering term by term all the conbinations of one even and one odd ensemble corresponding each to one error, and would number $r(90-r)$. We should have $r+r(90-r)=2^{11}$, which is impossible for integral values of r.

On the other side, the second case can be coded so as to yield 12 sure symbols, and the a_{mk} matrix of this case is given in Table I. A second matrix is also given, which is that of the only other lossless coding scheme encountered (in addition to the general class mentioned above) in which blocks of eleven ternary symbols are transmitted with no more than 2 errors, and out of which six sure symbols can be obtained.

It must be mentioned that the use of the ternary coding scheme just mentioned will always result in a power loss, whereas the coding scheme for 23 binary symbols and a maximum of three transmission errors yields a power saving of $1\frac{1}{2}$ db for vanishing probabilities of errors. The saving realized with the coding scheme for blocks of 2^n-1 binary symbols approaches 3 db for increasing n's and decreasing probabilities of error, but a loss is always encountered when $n=3$.

Marcel J. E. Golay
Signal Corps Engineering Laboratories
Fort Monmouth, N. J

TABLE I

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}
X_1	1	0	0	1	1	1	0	0	0	1	1	1
X_2	1	0	1	0	1	1	0	1	1	0	0	1
X_3	1	0	1	1	0	1	1	0	1	0	1	0
X_4	1	0	1	1	1	0	1	1	0	1	0	0
X_5	1	1	0	0	1	1	1	0	1	1	0	0
X_6	1	1	0	1	0	1	1	1	0	0	0	1
X_7	1	1	0	1	1	0	0	1	1	0	1	0
X_8	1	1	1	0	0	1	0	1	0	1	1	0
X_9	1	1	1	0	1	0	1	0	0	0	1	1
X_{10}	1	1	1	1	0	0	0	0	1	1	0	1
X_{11}	0	1	1	1	1	1	1	1	1	1	1	1

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
X_1	1	1	1	2	2	0
X_2	1	1	2	1	0	2
X_3	1	2	1	0	1	2
X_4	1	2	0	1	2	1
X_5	1	0	2	2	1	1

---

* Received by the Institute, February 23, 1949.

[1] C. E. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. Jour.*, vol. 27, p. 418; July, 1948.

Reprinted from *Proc. IRE,* vol. 37, p. 657, June 1949.

https://en.wikipedia.org/wiki/Marcel_J._E._Golay

What is a **code**? Computers use strings of bits, such as

$$\mathbf{010110111\cdots}.$$

You can't directly communicate them over long distance, because transmission errors might flip bits.

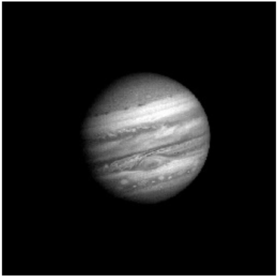
You need to add redundancies so that
small number of errors per bit can be corrected.

One encoding is the **Golay code**,
which **encodes original 12 bits into 24 bits**.

It is a particularly symmetric code: the symmetry is M_{24},
as noticed by Leech in 1967.

https://doi.org/10.4153/CJM-1967-017-0

It was actually used in the real world. One famous example is NASA's Voyager mission. Some of the scientific data from Jupiter was sent back using the Golay code.



This is the actual data of Jupiter from March 1979
which I took from the NASA website.

https://voyager.jpl.nasa.gov/mission/science/jupiter/

(If you read the documents from those days carefully, you find that the photographic image was *not* encoded by the Golay code, which was considered too wasteful.)

https://ntrs.nasa.gov/api/citations/19830002051/downloads/19830002051.pdf

Before talking about the **Mathieu Moonshine**,

I need to talk about the original **Monstrous Moonshine**.

Modular J function

$$J(q) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

is known from 19th century.

McKay noticed the following in **1978**:

The new finite simple group, the **Monster**,
which is the largest of the sporadics,
was being constructed at that time and has order $\sim 8 \cdot 10^{53}$.

The smallest nontrivial representation has dimension

$$196883$$

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The smallest nontrivial representation has dimension

$$\boldsymbol{196883}$$

Connects two distant branches of mathematics

$\boldsymbol{J}$ function : **classical complex analysis**
Monster group : **finite group**

Sounded too crazy back then, and called the **Monstrous Moonshine**.

(The word *moonshine* means foolish thought.)

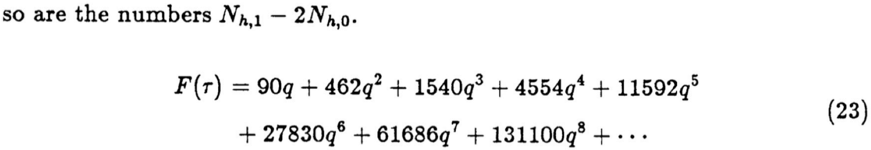
Mostly solved around the early 1990s
[Frenkel-Lepowsky-Meurman], [Borcherds]

and many developments since then.

The proof used ideas from **two-dimensional quantum field theories**.

I can finally come back to the **Mathieu Moonshine**.

In his PhD thesis, Ooguri computed the partition function of a **string moving in K3**:



so are the numbers $N_{h,1} - 2N_{h,0}$.

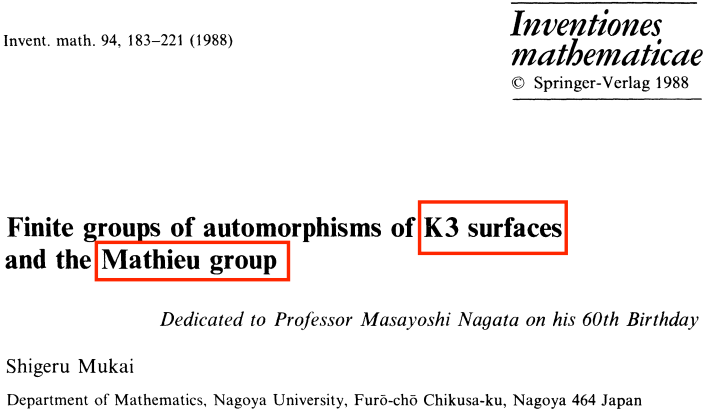
$$F(\tau) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + 61686q^7 + 131100q^8 + \cdots \tag{23}$$

This means that

- the first excited state has degeneracy **90**,
- the second excited state has degeneracy **462**,
- the third excited state has degeneracy **1540**, …

Around the same time, there was also the following paper:

http://eudml.org/doc/143625



Invent. math. 94, 183–221 (1988)

Inventiones mathematicae
© Springer-Verlag 1988

Finite groups of automorphisms of K3 surfaces and the Mathieu group

Dedicated to Professor Masayoshi Nagata on his 60th Birthday

Shigeru Mukai

Department of Mathematics, Nagoya University, Furō-chō Chikusa-ku, Nagoya 464 Japan

which says that the possible symmetries of K3 are certain small subgroups of the Mathieu group M_{24}.

Eguchi, his advisor, thought:

so are the numbers $N_{h,1} - 2N_{h,0}$.

$$F(\tau) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + 61686q^7 + 131100q^8 + \cdots \tag{23}$$

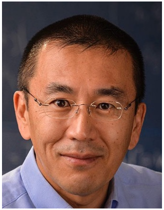
These coefficients should be related to Mathieu group.

And nothing happened for twenty years ...

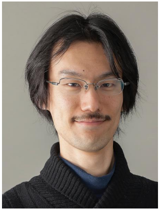
In the meantime, I became a student of Eguchi
obtained PhD in 2006, became a postdoc ...

T. Eguchi

H. Ooguri



me



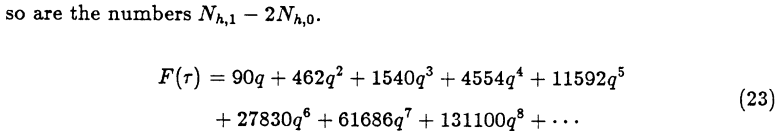
Aspen, Colorado, **Aug. 6th, 2009**.

All three were in the workshop.
We revisited the question.

I said:

**Why don’t we look up the table
in the Iwanami math encyclopedia?**

PhD thesis of Ooguri-san:



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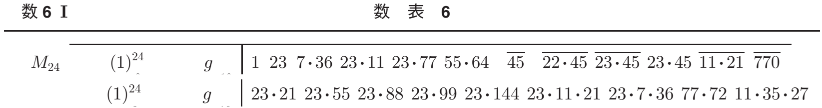
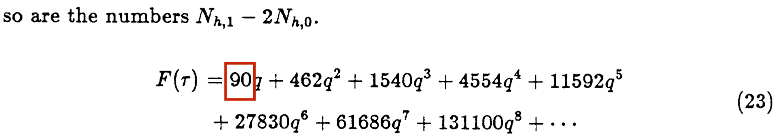
Iwanami Math Encyclopedia, 4th ed.:



数 6 I 数 表 **6**

M_{24}	$(1)^{24}$	g	$1\ \ 23\ \ 7\cdot 36\ \ 23\cdot 11\ \ 23\cdot 77\ \ 55\cdot 64\ \ \overline{45}\ \ \overline{22\cdot 45}\ \ \overline{23\cdot 45}\ \ 23\cdot 45\ \ \overline{11\cdot 21}\ \ \overline{770}$
	$(1)^{24}$	g	$23\cdot 21\ \ 23\cdot 55\ \ 23\cdot 88\ \ 23\cdot 99\ \ 23\cdot 144\ \ 23\cdot 11\cdot 21\ \ 23\cdot 7\cdot 36\ \ 77\cdot 72\ \ 11\cdot 35\cdot 27$

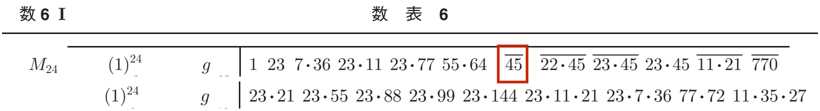
PhD thesis of Ooguri-san:



so are the numbers $N_{h,1} - 2N_{h,0}$.

$$F(\tau) = \boxed{90}q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + 61686q^7 + 131100q^8 + \cdots \qquad (23)$$

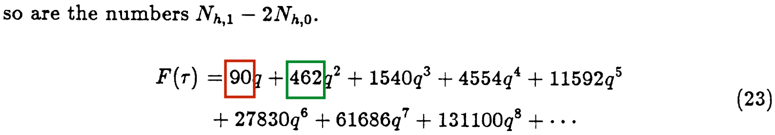
Iwanami Math Encyclopedia, 4th ed.:



数 6 I 数 表 6

M_{24}	$(1)^{24}$	g	1	23	7·36	23·11	23·77	55·64	$\boxed{\overline{45}}$	$\overline{22\cdot45}$	$\overline{23\cdot45}$	23·45	$\overline{11\cdot21}$	$\overline{770}$
	$(1)^{24}$	g	23·21	23·55	23·88	23·99	23·144	23·11·21	23·7·36	77·72	11·35·27			

PhD thesis of Ooguri-san:



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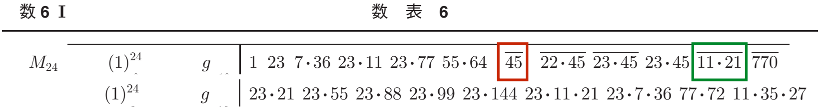
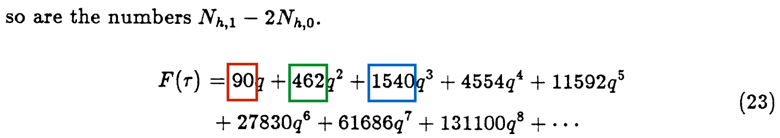


数 6 I　　　数 表 6

M_{24}	$(1)^{24}$	g	$1\ \ 23\ \ 7\cdot 36\ \ 23\cdot 11\ \ 23\cdot 77\ \ 55\cdot 64\ \ \boxed{\overline{45}}\ \ \overline{22\cdot 45}\ \ \overline{23\cdot 45}\ \ 23\cdot 45\ \ \boxed{\overline{11\cdot 21}}\ \ \overline{770}$
	$(1)^{24}$	g	$23\cdot 21\ \ 23\cdot 55\ \ 23\cdot 88\ \ 23\cdot 99\ \ 23\cdot 144\ \ 23\cdot 11\cdot 21\ \ 23\cdot 7\cdot 36\ \ 77\cdot 72\ \ 11\cdot 35\cdot 27$

PhD thesis of Ooguri-san:



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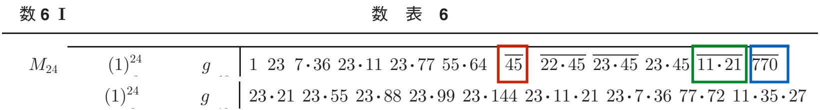
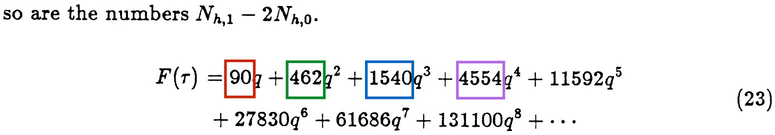
Iwanami Math Encyclopedia, 4th ed.:



数 6 I **数　表　6**

M_{24}	$(1)^{24}$	g	1	23	7·36	23·11	23·77	55·64	$\boxed{\overline{45}}$	$\overline{22\cdot45}$	$\overline{23\cdot45}$	23·45	$\boxed{\overline{11\cdot21}}$	$\boxed{\overline{770}}$
	$(1)^{24}$	g	23·21	23·55	23·88	23·99	23·144	23·11·21	23·7·36	77·72	11·35·27			

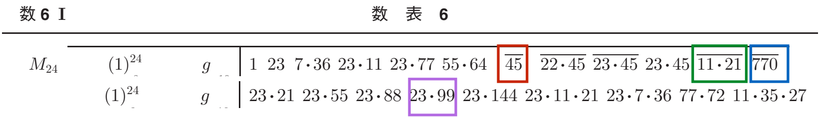
PhD thesis of Ooguri-san:



so are the numbers $N_{h,1} - 2N_{h,0}$.

$$F(\tau) = \boxed{90}q + \boxed{462}q^2 + \boxed{1540}q^3 + \boxed{4554}q^4 + 11592q^5 + 27830q^6 + 61686q^7 + 131100q^8 + \cdots \tag{23}$$

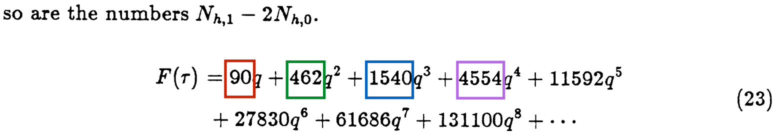
Iwanami Math Encyclopedia, 4th ed.:



数 6 I 　　　　　　数 表 6

M_{24}	$(1)^{24}$	g	1	23	$7\cdot36$	$23\cdot11$	$23\cdot77$	$55\cdot64$	$\boxed{\overline{45}}$	$\overline{22\cdot45}$	$\overline{23\cdot45}$	$23\cdot45$	$\boxed{\overline{11\cdot21}}$	$\boxed{\overline{770}}$
	$(1)^{24}$	g	$23\cdot21$	$23\cdot55$	$23\cdot88$	$\boxed{23\cdot99}$	$23\cdot144$	$23\cdot11\cdot21$	$23\cdot7\cdot36$	$77\cdot72$	$11\cdot35\cdot27$			

PhD thesis of Ooguri-san:



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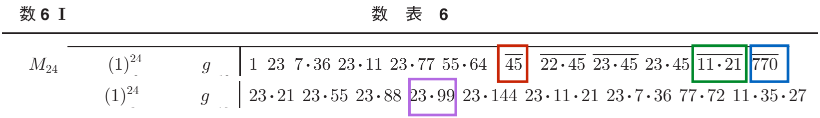
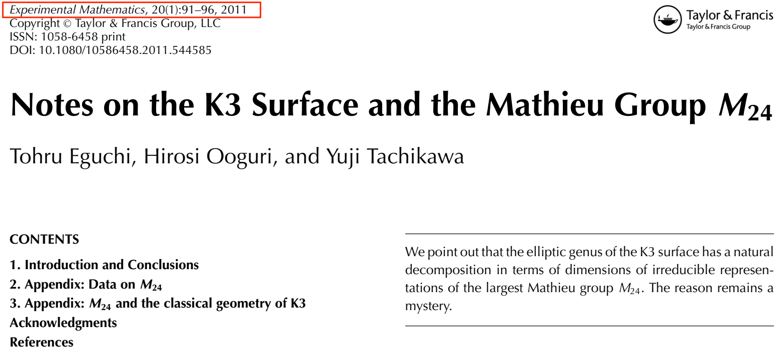


数 6 I　　数 表 6

M_{24}	$(1)^{24}$	g	1	23	7·36	23·11	23·77	55·64	$\boxed{\overline{45}}$	$\overline{22\cdot 45}$	$\overline{23\cdot 45}$	23·45	$\boxed{\overline{11\cdot 21}}$	$\boxed{\overline{770}}$
	$(1)^{24}$	g	23·21	23·55	23·88	$\boxed{23\cdot 99}$	23·144	23·11·21	23·7·36	77·72	11·35·27			

There is a correspondence!

We wrote a paper saying that there is a correspondence, and nothing more:



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Notes on the K3 Surface and the Mathieu Group M_{24}

Tohru Eguchi, Hirosi Ooguri, and Yuji Tachikawa

CONTENTS

We point out that the elliptic genus of the K3 surface has a natural decomposition in terms of dimensions of irreducible representations of the largest Mathieu group M_{24}. The reason remains a mystery.

https://doi.org/10.1080/10586458.2011.544585

https://arxiv.org/abs/1004.0956

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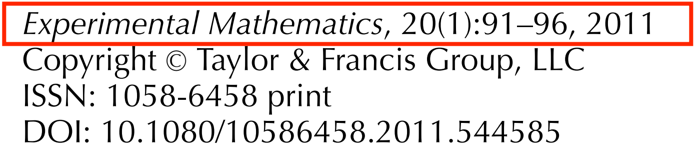

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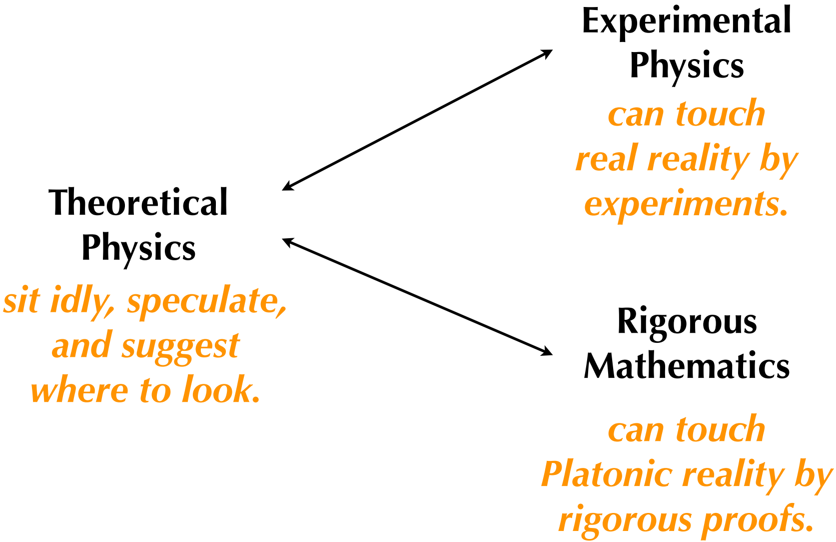
https://arxiv.org/abs/1004.0956

My contribution was literally only the suggestion
that we should look up the table.

Yes it was **essential**. But it was also totally **trivial**.

Eguchi and Ooguri *could have* looked up the same table in 1989.

This became one of the most cited papers of mine,
and both theoretical physicists and mathematicians still work on it.



Experimental Physics

can touch real reality by experiments.

Theoretical Physics

sit idly, speculate, and suggest where to look.

Rigorous Mathematics

can touch Platonic reality by rigorous proofs.