Mathematics of QFT, by QFT, for QFT

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What is **Quantum Field** Theory = QFT ?

- Describes **quantum** properties of **fields**, where
- **fields** are **anything which extend along space and time**, such as
- electromagnetic fields (=light), crystal vibration, electron fields …

Mathematics **of** QFT

Mathematics **by** QFT

Mathematics **for** QFT

Mathematics **of** QFT

=Math **to formulate** QFT

Mathematics **by** QFT

Mathematics **for** QFT

Mathematics **of** QFT
=Math **to formulate** QFT

Mathematics **by** QFT
=Math **inspired by** QFT

Mathematics **for** QFT

Mathematics **of** QFT
=Math **to formulate** QFT

Mathematics **by** QFT
=Math **inspired by** QFT

Mathematics **for** QFT
=Math **used to study** QFT

On terminology, 1.

Quantum field theory = field theory

場の量子論 = 場の理論

On terminology, 2.

There is a concept called a **field** in mathematics,
such as $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_q$

Quantum field is **not** the quantum version of that.

This distinction is clear in most other languages:

English	field	field
日本語	場	体
Deutsch	Feld	Körper
Fran caise	champs	corps

On terminology, 3.

The standard math usage is

- **Group**=a math object satisfying a set of axioms
- **Group theory**=Research on groups

So I think it should be that

- **Quantum field theory** =a math object satisfying a set of axioms
- **Quantum field theory theory**=Research on quantum field theory

but we refer to both of them as quantum field theory.

Before talking about QFT and Math,

let me tell you a bit about QFT and Physics.

The first successful example of **QFT** is
the **Quantum Electrodynamics** (QED, 量子電磁力学):

- describes quantized electromagnetic fields interacting with charged particles
- established around 1950s
- many developments since then

Theory and experiment match extremely well in QED.

The prime example is the **anomalous magnetic moment of electron**:

$$
\begin{aligned}
a_e^{\text{theory}} &= 0.01\,159\,652\,181\ldots \\
a_e^{\text{experiment}} &= 0.01\,159\,652\,181\ldots
\end{aligned}
$$

In QED, various quantities are computed as a formal power series in the fine structure constant (微細構造定数) $\alpha \sim 1/137$:

$$a_e = c_1 \left(\frac{\alpha}{\pi}\right) + c_2 \left(\frac{\alpha}{\pi}\right)^2 + c_3 \left(\frac{\alpha}{\pi}\right)^3 + \cdots$$

Coefficients c_1, c_2 are computed using Feynman diagrams, which describe multi-variable integrals of rational functions.

c_i	value	# of diagrams	Paper
c_1	$+0.5$	1	[Schwinger, 1948]
c_2	-0.3	7	[Sommerfield/Petermann, 1957-1958]
c_3	$+1.1$	72	[Laporta-Remiddi, hep-ph/9602417]
c_4	-1.9	891	[木下・仁尾, hep-ph/0507249]
c_5	$+9.2$	12672	[青山・早川・木下・仁尾, 1205.5368]

Remark 1.

The formal power series

$$a_e = c_1 \left(\frac{\alpha}{\pi}\right) + c_2 \left(\frac{\alpha}{\pi}\right)^2 + c_3 \left(\frac{\alpha}{\pi}\right)^3 + \cdots$$

does not converge and is asymptotic.

Physicists plug in $\alpha \sim 1/137$ anyway.

And it agrees very well with experiment.

Remark 2.

$c_{1,2,3}$ are known analytically:

$$
\begin{aligned}
c_1 &= \frac{1}{2}, \\
c_2 &= \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \log 2 + \frac{3}{4}\zeta(3), \\
c_3 &= \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) \\
&\quad + \frac{100}{3}\left[\mathrm{Li}_4(\frac{1}{2}) + \frac{1}{24}(\log 2)^4 - \frac{1}{24}\pi^2(\log 2)^2\right] \\
&\quad - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2(\log 2) + \frac{17101}{810}\pi^2 + \frac{28259}{5184}.
\end{aligned}
$$

Remakr 3. Feynman diagrams contributing to c_5 look like:

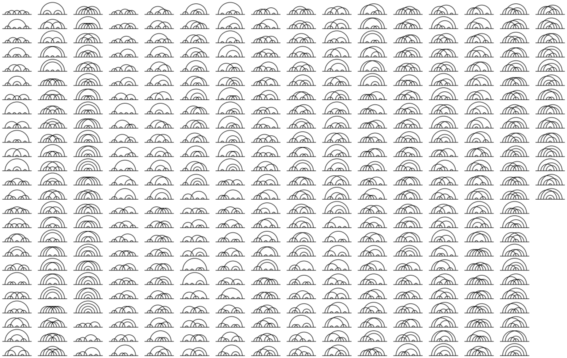


Figure: 389 self-energy diagrams representing 6354 vertex diagrams of Set V.

[http://www.riken.jp/lab-www/theory/colloquium/kinoshita.pdf]
Taken from the colloquium by Prof. Kinoshita. Highly recommended.

Quantum chromodynamics (QCD, 量子色力学) is about the force binding three quarks into protons (陽子) and neutrons (中性子).

The Taylor series can be computed as in QED,
but plugging $\alpha_{\mathrm{QCD}} \sim 1$ does not work.

Instead, people simulate the theory on supercomputers.

Called **Lattice QCD** since the spacetime is approximated by a lattice.

Has long history, and finally in the last ten years
the simulated results agree fairly well with experiments
thanks to the increase in computing power & computational techniques.

Tsukuba U.(筑波大) is one of the centers of this study in the world!

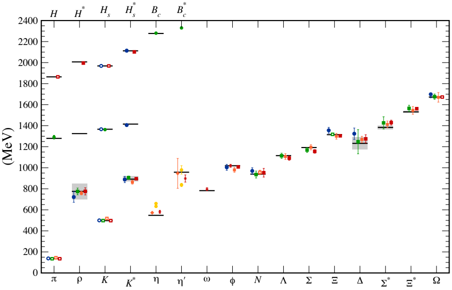


Figure 2: Hadron spectrum from lattice QCD. Comprehensive results for mesons and baryons are from MILC (27, 28), PACS-CS (29), BMW (30), and QCDSF (31). Results for η and η' are from RBC & UKQCD (32), Hadron Spectrum (33) (also the only ω mass), and UKQCD (34). Results for heavy-light hadrons from Fermilab-MILC (35), HPQCD (36), and Mohler & Woloshyn (37). Circles, squares, and diamonds stand for staggered, Wilson, and chiral sea quarks, respectively. Asterisks represent anisotropic lattices. Open symbols denote the masses used to fix parameters. Filled symbols (and asterisks) denote results. Red, orange, yellow, green, and blue stand for increasing numbers of ensembles (i.e., lattice spacing and sea quark mass). Horizontal bars (gray boxes) denote experimentally measured masses (widths). b-flavored meson masses are offset by -4000 MeV.

Taken from [Kronfeld 1203.1204]

So far I only talked about the particle physics (素粒子物理).

QFT is also used in condensed matter physics (物性物理).

Conformal field theory(共形場理論) describes
2nd order phase transitions and has a long history.

Recently popular **topological condensed-matter physics**
also uses QFT a lot.

Summarizing,

- QFT is well researched,
- QFT predicts quantities with high precision, and
- QFT agrees very well with experiment.

But it is **mathematically incomplete**, in the following sense.

Quantum mechanics and general relativity can be described to mathematicians **in a sentence**.

Quantum mechanics is

the study of unitary operators on Hilbert spaces.

General relativity is

the study of the Einstein equation on manifolds.

QFT is

???

As I've been doing QFT for a long time,
I think I know.
But still I cannot describe it in a sentence.

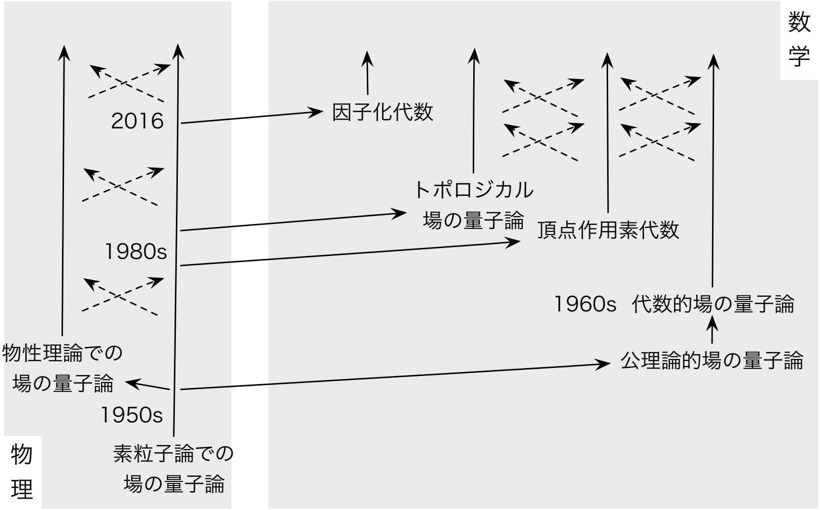
Of course it's OK if you can't describe it mathematically in a sentence.

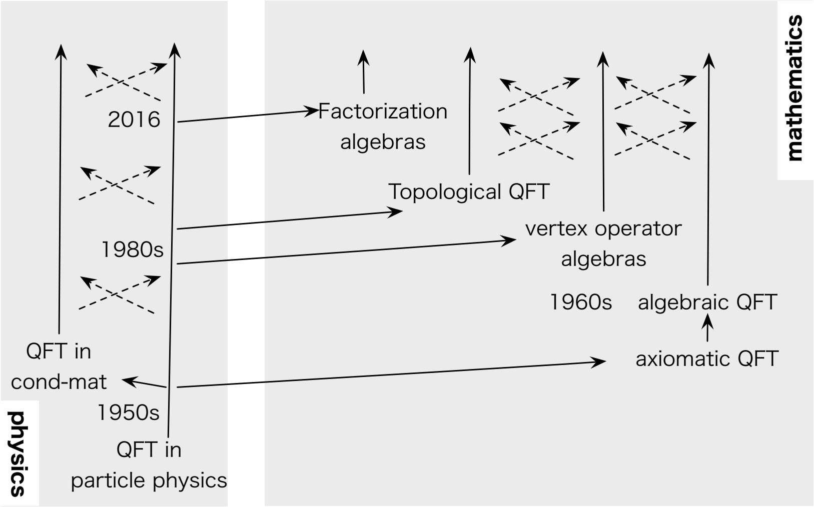
Ancient Egyptians could build pyramids,
although they had not formalized geometry.

Physicists can compute things although they have not formalized QFT.

But I feel the situation unsatisfactory.

Many had the same feeling, and tried to formulate QFT:


数学
物理
因子化代数
2016
トポロジカル
場の量子論
頂点作用素代数
1980s
1960s
代数的場の量子論
公理論的場の量子論
物性理論での
場の量子論
1950s
素粒子論での
場の量子論


mathematics
Factorization
algebras
2016
Topological QFT
vertex operator
algebras
1980s
1960s
algebraic QFT
axiomatic QFT
QFT in
cond-mat
1950s
physics
QFT in
particle physics

That said, the computation of a_e the anomalous magnetic moment of electron, **cannot** be presented in any of

- topological QFT(トポロジカル場の理論),
- vertex operator algebras (頂点作用素代数), or
- axiomatic/algebraic QFTs (公理論的/代数的場の理論).

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QFT is

???

(Factorization algebras of Costello-Gwilliam does allow us to express the computation of a_e. But that is not the only criterion.)

Math of QFT

$\downarrow$

Math for QFT

$\downarrow$

Math by QFT

Math of QFT

$\downarrow$

Math for QFT

$\downarrow$

Math by QFT

QFT is mathematically incomplete.

But the research on QFT requires math.

I've been always learning some new subfield of math.

For example...

As a grad student :	Sasaki-Einstein manifolds
As a postdoc:	nilpotent orbits
Recently :	algebraic topology, mostly about cobordisms

(It's not that all QFT people needed to learn them.
I guess my situation is not typical.
It so happened that my type of QFT study somehow needed them.)

I recently needed to learn **Anderson dual of generalized (co)homology**.

Ordinary cohomology has the universal coefficient theorem:

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H_{d-1}(X,\mathbb{Z}),\mathbb{Z}) \to H^d(X,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_d(X,\mathbb{Z}),\mathbb{Z}) \to 0.$$

For a generalized (co)homology theory $h_*(-)$, $h^*(-)$,
there is the Anderson dual $Dh_*(-)$, $Dh^*(-)$
s.t. it satisfies the same universal coefficient theorem:

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(h_{d-1}(X), \mathbb{Z}) \to (Dh)^d(X) \to \mathrm{Hom}_{\mathbb{Z}}(h_d(X), \mathbb{Z}) \to 0$$

And $DDh = h$.

The universal coefficient theorem of $H(-, \mathbb{Z})$ means that
$DH(-, \mathbb{Z}) = H(-, \mathbb{Z})$.

Similarly, $DK = K$, $DKO = KSp$.

Why did I need this?

Topological materials or equivalently
topological condensed matter physics is a very hot topic these days.

QFTs called 対称性保護トポロジカル相 = **SPT 相**
=**(Symmetry protected topological phases**= **SPT phase**
play important roles.

People realized that their classification is given by
the **Anderson dual of bordism homology theories**:
[Kapustin-Thorngren-Turzillo-Wang 1406.7329],
[Freed-Hopkins 1604.06527], [Yonekura 1803.10796]

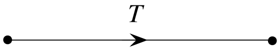
I'd like to explain this more.

For this, I need to explain to you what is QFT, at least to some extent.

Let us consider **Quantum Mechanics** (量子力学) to start with.

We have the Hilbert space $\mathcal{H}$.

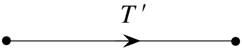
The time evolution by T



gives a linear operator

$$U(T) : \mathcal{H} \to \mathcal{H}.$$

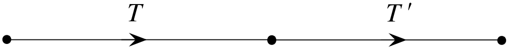
Similarly,



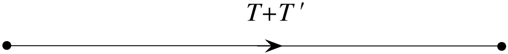
gives

$$U(T') : \mathcal{H} \to \mathcal{H}$$

Since



and



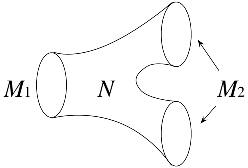
are the same, we require

$$U(T)U(T') = U(T + T').$$

Now it is easy to generalize it to d-**dimensional QFT**:
For $d-1$ dimensional manifold M, it assigns a Hilbert space

$$\mathcal{H}(M),$$

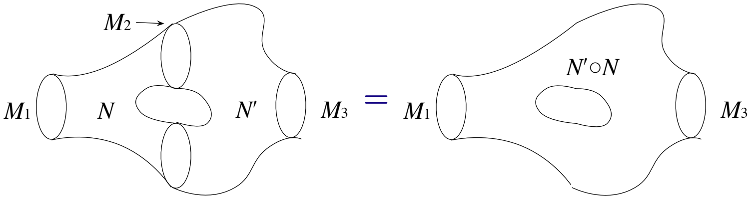
and for



it assigns

$$U(N) : \mathcal{H}(M_1) \to \mathcal{H}(M_2).$$

Corresponding to



we require

$$U(N')U(N) = U(N' \circ N).$$

Remark 1.

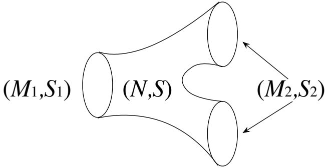
When we say “manifolds”,
we need to specify how much structure one wants to put on them.

Smooth structure only = **topological QFT**.

Smooth structure + metric + spin structure
= **QFT usually encountered in physics**.

Then d-dimensional $\mathcal{S}$-structured QFT Q contains,
as a tiny part of the data, a **functor**

- from the category of $d-1$-dimensional $\mathcal{S}$-structured manifolds and bordisms between them
- to the category of Hilbert spaces and linear maps.

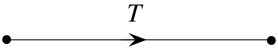


$$\Longrightarrow U_Q(N,S) : \mathcal{H}(M_1,S_1) \to \mathcal{H}(M_2,S_2)$$

where S is a $\mathcal{S}$ structure on N, etc.

Remark 2.

Quantum Mechanics assigns to



the linear map

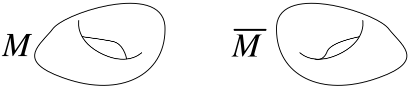
$$U(t) : \mathcal{H} \to \mathcal{H}.$$

By regarding $\mathcal{H} = \mathcal{H}(\bullet)$, this is equivalent to
1-dimensional QFT for manifolds with metric.

Remark 3.

In physics, we need positive-definite metric on $\mathcal{H}$,
since the norm of a vector is related to probability.

There are relations such as



implies

$$\overline{\mathcal{H}(M)} = \mathcal{H}(\overline{M}).$$

Called **unitary QFTs** = **reflection-positive QFTs**

Remark 4.

Take Q_1, Q_2: d-dimensional $\mathcal{S}$-structured QFTs.

Then we can consider their direct sum $Q_1 + Q_2$ and
the direct product $Q_1 \times Q_2$ such that
for $d-1$ dimensional $\mathcal{S}$-structured manifold M
we have

$$\mathcal{H}_{Q_1+Q_2}(M) = \mathcal{H}_{Q_1}(M) \oplus \mathcal{H}_{Q_2}(M),$$
$$\mathcal{H}_{Q_1\times Q_2}(M) = \mathcal{H}_{Q_1}(M) \otimes \mathcal{H}_{Q_2}(M).$$

The 'set' of QFTs has the operations $+$ and $\times$.

When $\mathcal{H}_Q(M)$ is one-dimensional for all M,
such a Q is called **invertible**.

Invertible QFTs are invertible elements in the 'algebra' formed by QFTs under $Q_1 + Q_2$, $Q_1 \times Q_2$: you just take

$$\mathcal{H}_{Q^{-1}}(M) := \mathcal{H}_Q(M)^\vee.$$

Invertible QFTs form an Abelian group under $\times$.

If one wants to study QFT mathematically,
you need to understand them first.

But they were not at all studied systematically until around 2010.

Why? My guess is the following:

We often say **QFTs are difficult because of infinities.**

Even the Hilbert space of one quantum-mechanical particle, $L^2(\mathbb{R})$, is infinite dimensional.

Qubit, which are important in quantum information theory, is already two-dimensional.

But **Hilbert space of invertible QFTs are one-dimensional.**

One dimensional Hilbert spaces sounded too simple.

The reason why invertible QFTs got attention since around 2010 is that

unitary invertible QFTs =
対称性保護トポロジカル相 =
symmetry protected topological phase

Given a structure $\mathcal{S}$ on manifolds M,
let a new structure $\mathcal{S}(X)$ for a space X
= an $\mathcal{S}$ structure on M + a map $M \to X$.

Classification of $\mathcal{S}(X)$-structured d-dimensional invertible QFTs up to continuous change

=

Classification of X-protected topological phases

is a big problem in condensed matter physics.

($\mathcal{S}$ is usually not mentioned in cond-mat and it is implicitly taken to be something standard such as smooth structure + spin structure + metric)

Answer: Classification of $\mathcal{S}(X)$-structured d-dimensional invertible QFTs up to continuous change is given by

$$(D\Omega^{\mathcal{S}})^{d+1}(X)$$

where $\Omega^{\mathcal{S}}_{\bullet}$ is the $\mathcal{S}$-structured bordism homology and D is the Anderson dual.

[Kapustin-Thorngren-Turzillo-Wang 1406.7329],
[Freed-Hopkins 1604.06527],
[Yonekura 1803.10796]

Why Anderson dual?

In a real experiment, people deal with finite-sized samples.

As an idealization, consider an experimental sample filling the entire universe of the form of a closed manifold N_d.

Considering N_d as a bordism from $\varnothing$ to $\varnothing$, we can regard

$$U_Q(N_d) : \mathcal{H}(\varnothing) \to \mathcal{H}(\varnothing)$$

as

$$U_Q(N_d) : \mathbb{C} \to \mathbb{C}$$

and therefore it gives a complex number, called the partition function.

Therefore Q gives rise to the function

$$N_d \mapsto \frac{U_Q(N_d)}{|U_Q(N_d)|} \in \{\text{number of absolute value one}\}$$

which is defined more or less on d-dimensional bordisms.

So it gives some kind of dual to the bodrism homology.

More detailed analysis tells you the correct dual to use here is the Anderson dual.

For example, **Time-reversal invariant topological superconductor**
= **時間反転不変トポロジカル超伝導体**
corresponds to the structure $\boldsymbol{\mathcal{S}}$ being Pin^{+}, and $\boldsymbol{X} = \bullet$ (a single point).

Here, Pin^{+} is a double cover of O just as
Spin group is a double cover of SO

Q. Classify time-reversal invariant top. superconductor in 4d.

[Kirby-Taylor "Pin structures on low-dimensional manifolds", London Math. Soc. LNS 151, pp.177-242] from 1990 says ...

$$\beta : \Omega_2^{Pin^-} \longrightarrow \mathbf{Z}/8\mathbf{Z}$$

gives the isomorphism in the following table.

$\Omega_1^{Spin} = \mathbf{Z}/2\mathbf{Z}$	$\Omega_2^{Spin} = \mathbf{Z}/2\mathbf{Z}$	$\Omega_3^{Spin} = 0$	$\Omega_4^{Spin} = \mathbf{Z}$
$\Omega_1^{Pin^-} = \mathbf{Z}/2\mathbf{Z}$	$\Omega_2^{Pin^-} = \mathbf{Z}/8\mathbf{Z}$	$\Omega_3^{Pin^-} = 0$	$\Omega_4^{Pin^-} = 0$
$\Omega_1^{Pin^+} = 0$	$\Omega_2^{Pin^+} = \mathbf{Z}/2\mathbf{Z}$	$\Omega_3^{Pin^+} = \mathbf{Z}/2\mathbf{Z}$	$\Omega_4^{Pin^+} = \mathbf{Z}/16\mathbf{Z}$

In §2 we calculate the 1 and 2 dimensional groups and show that the non–zero one dimensional groups are generated by the circle with its Lie group framing, S^1

It also says $\Omega_{4+1}^{\text{pin}+}(\bullet) = 0$. Therefore

$$(D\Omega^{\text{pin}+})^{4+1}(\bullet) = \mathbb{Z}_{16}.$$

Time-reversal invariant top. superconductor in 4d is classified by $\mathbb{Z}_{\mathbf{16}}$!

$$\rho : \Omega_2 \longrightarrow \mathbf{Z}/8\mathbf{Z}$$

gives the isomorphism in the following table.

$$\begin{array}{llll}
\Omega_1^{Spin} = \mathbf{Z}/2\mathbf{Z} & \Omega_2^{Spin} = \mathbf{Z}/2\mathbf{Z} & \Omega_3^{Spin} = 0 & \Omega_4^{Spin} = \mathbf{Z} \\
\Omega_1^{Pin^-} = \mathbf{Z}/2\mathbf{Z} & \Omega_2^{Pin^-} = \mathbf{Z}/8\mathbf{Z} & \Omega_3^{Pin^-} = 0 & \Omega_4^{Pin^-} = 0 \\
\Omega_1^{Pin^+} = 0 & \Omega_2^{Pin^+} = \mathbf{Z}/2\mathbf{Z} & \Omega_3^{Pin^+} = \mathbf{Z}/2\mathbf{Z} & \Omega_4^{Pin^+} = \mathbf{Z}/16\mathbf{Z}
\end{array}$$

In §2 we calculate the 1 and 2 dimensional groups and show that the non–zero one dimensional groups are generated by the circle with its Lie group framing, S^1_{Lie},

This matches with the results condensed-matter physicists originally obtained with physics intuition.

When Kirby-Taylor computed $\boldsymbol{\Omega_4^{\text{pin}+} = \mathbb{Z}_{16}}$,
they did not care about topological condensed-matter phases.

However, thanks to Kirby-Taylor who computed them
and presented them clearly,
we can just use their results.

My 15-year+ experience of QFT tells me that
once we successfully reduce a QFT question to a math question,
mathematicians often already solved it independently
so we can just borrow the results.

Thank you, mathematicians !

The good thing about textbooks / papers by mathematicians is:

the definition and the theorem are stated carefully,
so that they can be used as tools
even without reading/understanding the proofs.

It's the same attitude where I touch the icon of apps
in a smartphone, without understanding why it works.

Textbooks / papers by physicists are terrible in comparison.

It's very hard to tell what was assumed and what was derived;
you are forced to read everything.

Math of QFT

$\downarrow$

Math for QFT

$\downarrow$

Math by QFT

Math of QFT

$\downarrow$

Math for QFT

$\downarrow$

Math by QFT

Many subjects of math arose, inspired by results in QFT.

Two most famous examples are:

- Mirror symmetry of Calabi-Yau manifolds

- Seiberg-Witten theory of four-dimensional manifolds

Many subjects of math arose, inspired by results in QFT.

Two most famous examples are:

- Mirror symmetry of Calabi-Yau manifolds
  $\Longleftarrow$ came from **2d supersymmetric QFT**

- Seiberg-Witten theory of four-dimensional manifolds

Many subjects of math arose, inspired by results in QFT.

Two most famous examples are:

- Mirror symmetry of Calabi-Yau manifolds
  $\Longleftarrow$ came from **2d** **supersymmetric QFT**

- Seiberg-Witten theory of four-dimensional manifolds
  $\Longleftarrow$ came from **4d** **supersymmetric QFT**

Supersymmetric QFT?

In the terminology of this talk, it is a specific choice of a structure $\mathcal{S}$ which we put on manifolds.

Only exists below $d \leq 11$ dimensions,
and depends in detail on specific choice of d.

This makes it hard to explain to mathematicians.

I guess I need to think more how to present it better.

Two-dimensional (n, n') supersymmetric structure has:

smooth structure + spin structure + metric
+ principal $\mathbf{SO}(n) \times \mathbf{SO}(n')$ with connection
+ many more.

Given a $(2,2)$ supersymmetric structure on 2d manifold N

$$S = (\text{smooth structure}, \text{metric}, \mathbf{SO(2)}\ \text{bundle}\mathbf{1}, \mathbf{SO(2)}\ \textbf{bundle2}, \ldots)$$

we define another $\mathbf{SO(2)}$ bundle
using the outer morphism $\mathbf{SO(2)} \subset \mathbf{O(2)}$
which gives a new $(2,2)$ supersymmetric structure S° :

$$S^\circ := (\text{smooth structure}, \text{metric}, \mathbf{SO(2)}\ \text{bundle}\mathbf{1}, \mathbf{SO(2)}\ \textbf{bundle2}', \ldots).$$

We have

$$S^{\circ\circ} = S.$$

Correspondingly, two-dimensional $(2,2)$ supersymmetric QFT Q gives rise to Q° via :

$$\mathcal{H}_{Q^\circ}(M,S) := \mathcal{H}_Q(M,S^\circ), \qquad U_{Q^\circ}(N,S) := U_Q(N,S^\circ).$$

We also have $Q^{\circ\circ} = Q$.

Q° is called the mirror of Q.

For a two-dimensional $\mathbf{(2,2)}$ supersymmetric QFT $\boldsymbol{Q}$, we can extract

- Triangulated category $\boldsymbol{A(Q)}$ formed by A-type boundary conditions,
- Triangulated category $\boldsymbol{B(Q)}$ formed by B-type boundary conditions.

Almost by definition,

$$\boldsymbol{A(Q^\circ) = B(Q), \qquad B(Q^\circ) = A(Q).}$$

Given a Calabi-Yau variety X,
we can construct 2d $(2, 2)$ supersymmetric QFT $Q(X)$. Then

- $A(Q(X))$ is the Fukaya category of X;
- $B(Q(X))$ is the derived category of coherent sheaves of X.

It often happens that for two Calabi-Yau varieties X, Y we have

$$Q(X)^\circ = Q(Y).$$

Then

$$A(Q(X)) = B(Q(Y)), \qquad A(Q(Y)) = B(Q(X))$$

therefore

- Fukaya category of X = derived category of coherent sheaves of Y
- Fukaya category of Y = derived category of coherent sheaves of X

Homological mirror symmetry, from the mid 1990s.

Remark.

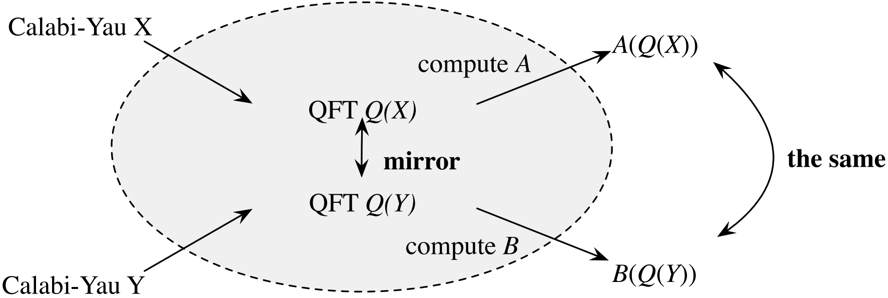
For a given Calabi-Yau manifold X and
the corresponding 2d (2,2) supersymmetric QFT $Q(X)$,
it often happens that

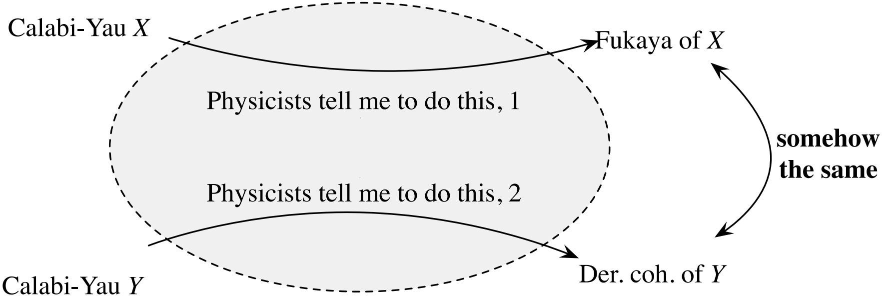
$$Q(X)^{\circ} = Q(Y)$$

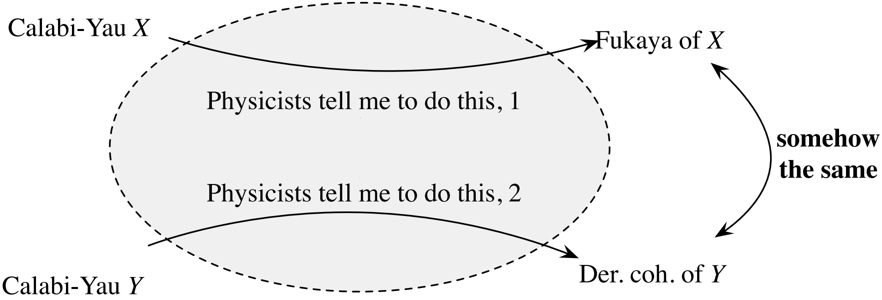
for no choice of Calabi-Yau Y.

This is the case e.g. when X does not have
any complex structure deformation.

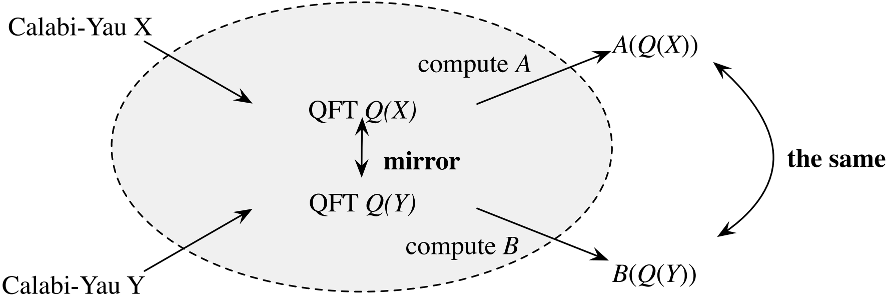
This is mainly because there are many 2d (2,2) supersymmetric QFTs
not of the form $Q(X)$.


Calabi-Yau X
A(Q(X))
compute A
QFT Q(X)
mirror
the same
QFT Q(Y)
compute B
B(Q(Y))
Calabi-Yau Y


Calabi-Yau X
Fukaya of X
Physicists tell me to do this, 1
somehow
the same
Physicists tell me to do this, 2
Der. coh. of Y
Calabi-Yau Y



Things shaded gray are not well defined. But there are some logic.



Things shaded gray are not well defined. But there are some logic.

Three dimensional n-extended supersymmetric structure contains:

smooth structure + spin structure + metric
+ principal $\mathbf{SO}(n)$ bundle with connection
+ many more.

Given a 4-extended supersymmetric structure on N,

$$S = (\text{smooth structure}, \text{metric}, \mathrm{SO}(4)\ \textbf{bundle}, \ldots)$$

we apply the outer automorphism
coming from $\mathbf{SO(4)} \subset \mathbf{O(4)}$ on the $\mathbf{SO(4)}$ bundle,
which gives a new $\mathbf{4}$-extended supersymmetric structure S°:

$$S^\circ := (\text{smooth structure}, \text{metric}, \mathrm{SO}(4)\ \textbf{bundle}', \ldots).$$

We have

$$S^{\circ\circ} = S.$$

Correspondingly, 3d $\mathbf{4}$-extended QFT Q gives rise to Q° :

$$\mathcal{H}_{Q^\circ}(M,S) := \mathcal{H}_Q(M,S^\circ), \qquad U_{Q^\circ}(N,S) := U_Q(N,S^\circ).$$

We have $Q^{\circ\circ} = Q$.

Q° is called the mirror of Q.

Given a 3d $\mathbf{4}$-extended supersymmetric QFT $\boldsymbol{Q}$, we can extract two hyperkähler manifolds

- $\mathbf{Higgs}(\boldsymbol{Q})$ called the Higgs branch
- $\mathbf{Coulomb}(\boldsymbol{Q})$ called the Coulomb branch

Almost by definition, we have

$$\mathbf{Higgs}(\boldsymbol{Q}^\circ) = \mathbf{Coulomb}(\boldsymbol{Q}), \qquad \mathbf{Coulomb}(\boldsymbol{Q}^\circ) = \mathbf{Higgs}(\boldsymbol{Q}).$$

Given a compact Lie group G and its quaternionic representation V we can construct 3d 4-extended supersymmetric QFT $Q(G, V)$. Then

$$\mathbf{Higgs}(Q(G, V)) = V///G$$

is the hyperkähler quotient of V by G, introduced in the late 1980s. Namely, there is the moment map

$$\mu : V \to \mathfrak{g}^* \otimes \mathbb{R}^3$$

and we define

$$V///G = \mu^{-1}(0)/G.$$

How about $\mathbf{Coulomb}(\boldsymbol{Q}(\boldsymbol{G}, \boldsymbol{V}))$?

This should also be hyperkähler, but was not defined mathematically until very recently.

Only a few years ago, [Braverman-Finkelberg-Nakajima] defined it as a holomorphic symplectic manifold.

They first defined a complex of constructible sheaves
on the affine Grassmannian of $\boldsymbol{G}$ using $\boldsymbol{V}$.
Then $\mathbf{Coulomb}(\boldsymbol{Q}(\boldsymbol{G}, \boldsymbol{V}))$ is the $\mathbf{Spec}$ of its cohomology ring.

If $Q(V, G)^\circ = Q(V', G')$, we have

$$\begin{aligned} \mathrm{Higgs}(Q(V,G)) &= \mathrm{Coulomb}(Q(V',G')), \\ \mathrm{Coulomb}(Q(V,G)) &= \mathrm{Higgs}(Q(V',G')) \end{aligned}$$

which then exchanges the hyperkähler quotient construction and the construction of B-F-N.

Often, for a given (V, G), there's **no** (V', G') such that

$$Q(V,G)^\circ = Q(V',G').$$

This is due to the fact that most of the three-dimensional 4-extended supersymmetric QFTs are not of the form $Q(V, G)$.

$\mathrm{Coulomb}(Q(V,G))$ gives many new types of hyperkähler manifolds.

[Atiyah-Drinfeld-Hitchin-Manin] realized
the moduli space of instantons of classical group K on $\mathbb{R}^4$ as a **hyperkähler quotient** for a suitable V, G:

$$\mathcal{M}_K = \mathrm{Higgs}(Q(V,G)) = V///G$$

Here, instantons of a group K = principal K-bundle with connection such that $F = -\star F$.

When K is **exceptional**,
physicists think that there's no (V,G) such that

$$\mathcal{M}_K = \mathrm{Higgs}(Q(V,G)) = V///G.$$

Physicists also think that $\mathcal{M}_K$ when K **is simply-laced** ,i.e. for $K =$

$$A_{n-1} = \mathrm{SU}(n), \qquad D_n = \mathrm{SO}(2n), \qquad E_{6,7,8}$$

there is a suitable choice of (V, G) such that

$$\mathcal{M}_K = \mathrm{Coulomb}(Q(V, G))$$

Recent definition of $\mathrm{Coulomb}(Q(V, G))$ brought us one step closer to the understanding of exceptional instanton moduli spaces.

Let me end with another story, this time rather personal.

Classification of finite simple groups

- Alternating group $A_{n\geq 5}$
  Infinitely many.
- Simple finite groups of Lie type
  $\simeq$ Lie groups considered on finite fields $\mathbb{F}_q$
  Infinitely many.
- Sporadic simple groups
  Just **26** of them:
  Mathieu groups, Janco groups, ..., **Monster group**.

Modular J function

$$J(q) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

is known from 19th century.

McKay noticed the following in **1978**:

The new finite simple group, **Monster** (モンスター)
which was being constructed at that time has order $\sim 8 \cdot 10^{53}$.

The smallest nontrivial representation has dimension

$$196883$$

Connects two distant branches of mathematics

J function :	**classical complex analysis**
Monster group :	**finite group**

Many related observations [Ogg, Conway-Norton, ...]

Sounded too crazy back then, and called the
Monstrous Moonshine (= とんでもなく馬鹿げた話)

Mostly solved at the early 1990s
[Frenkel-Lepowsky-Meurman], [Borcherds]

and many developments since then,

done by many people in this room.

Uses vertex operator algebras.

Related to the string theory on

$$(\mathbb{R}^{24}/(\text{Leech lattice}))/\mathbb{Z}_2.$$

Cast of the rest of the story:

江口さん	Eguchi-san	born 1948	
大栗さん	Ooguri-san	born 1962	student of Eguchi san
僕	me	born 1979	student of Eguchi san

In **1989**, Ooguri-san computed the spectrum of a string moving in the **K3** surface.

so are the numbers $N_{h,1} - 2N_{h,0}$.

$$F(\tau) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + 61686q^7 + 131100q^8 + \cdots \tag{23}$$

Is there a similar moonshine here?

Around the same time, there was also the following paper:

Invent. math. 94, 183–221 (1988)

Inventiones mathematicae

Finite groups of automorphisms of K3 surfaces and the Mathieu group

Dedicated to Professor Masayoshi Nagata on his 60th Birthday

Shigeru Mukai

Department of Mathematics, Nagoya University, Furō-chō Chikusa-ku, Nagoya 464 Japan

Eguchi-san thought

so are the numbers $N_{h,1} - 2N_{h,0}$.

$$F(\tau) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + 61686q^7 + 131100q^8 + \cdots \tag{23}$$

These coefficients should be related to Mathieu group.

Twenty years later...

Aspen, Colorado, **2009**.

Eguchi-san, Ooguri-san and I were in the same workshop.

I said:

Why don't we look up the table
in the Iwanami math encyclopedia = 岩波数学辞典?

PhD thesis of Ooguri-san:

so are the numbers $N_{h,1} - 2N_{h,0}$.

$$F(\tau) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + 61686q^7 + 131100q^8 + \cdots \tag{23}$$

Iwanami Math Encyclopedia, 4th ed.:

数 **6** I　　　数　表　**6**

M_{24}	$(1)^{24}$	g	$1\ \ 23\ \ 7\cdot 36\ \ 23\cdot 11\ \ 23\cdot 77\ \ 55\cdot 64\ \ \overline{45}\ \ \overline{22\cdot 45}\ \ \overline{23\cdot 45}\ \ 23\cdot 45\ \ \overline{11\cdot 21}\ \ \overline{770}$
	$(1)^{24}$	g	$23\cdot 21\ \ 23\cdot 55\ \ 23\cdot 88\ \ 23\cdot 99\ \ 23\cdot 144\ \ 23\cdot 11\cdot 21\ \ 23\cdot 7\cdot 36\ \ 77\cdot 72\ \ 11\cdot 35\cdot 27$

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数 6 I　　　　数　表　6

M_{24}	$(1)^{24}$	g	1 23 $7\cdot36$ $23\cdot11$ $23\cdot77$ $55\cdot64$ $\overline{45}$ $\overline{22\cdot45}$ $\overline{23\cdot45}$ $23\cdot45$ $\overline{11\cdot21}$ $\overline{770}$
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	$(1)^{24}$	g	23·21　23·55　23·88　23·99　23·144　23·11·21　23·7·36　77·72　11·35·27

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	$(1)^{24}$	g	$23\cdot21$	$23\cdot55$	$23\cdot88$	$23\cdot99$	$23\cdot144$	$23\cdot11\cdot21$	$23\cdot7\cdot36$	$77\cdot72$	$11\cdot35\cdot27$				

There is a correspondence!

We wrote a paper saying that there is a correspondence, and nothing more:

Notes on the K3 Surface and the Mathieu Group M_{24}

Tohru Eguchi, Hirosi Ooguri, and Yuji Tachikawa

Experimental Mathematics, 20(1):91–96, 2011

ISSN: 1058-6458 print
DOI: 10.1080/10586458.2011.544585

https://arxiv.org/abs/1004.0956

Steady progress since then, such as

arXiv.org > math > arXiv:1211.5531

Search or

Mathematics > Representation Theory

Much ado about Mathieu

Terry Gannon

(Submitted on 23 Nov 2012 (v1), last revised 15 Mar 2013 (this version, v2))

Eguchi, Ooguri and Tachikawa have observed that the elliptic genus of type II string theory on K3 surfaces appears to possess a Moonshine for the largest Mathieu group. Subsequent work by several people established a candidate for the elliptic genus twisted by each element of M24. In this paper we prove that the resulting sequence of class functions are true characters of M24, proving the Eguchi-Ooguri-Tachikawa conjecture. We prove the evenness property of the multiplicities, as conjectured by several authors. We also identify the role group cohomology plays in both K3-Mathieu Moonshine and Monstrous Moonshine; in particular this gives a cohomological interpretation for the non-Fricke elements in Norton's Generalised Monstrous Moonshine conjecture. We investigate the intriguing proposal of Gaberdiel-Hohenegger-Volpato that K3-Mathieu Moonshine lifts to the Conway group Co1.

https://arxiv.org/abs/1211.5531

But no VOA with M_{24} symmetry with the right property has been found.

Mathematics **of** QFT
=Math **to formulate** QFT

Mathematics **by** QFT
=Math **inspired by** QFT

Mathematics **for** QFT
=Math **used to study** QFT