On homomorphisms from finite subgroups of $SU(2)$ to Langlands pairs of groups

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math paper: [arXiv:2505.01253] with Yuki Kojima
physics paper: to appear with Sunjin Choi

virtually at SIMIS, May 28, 2025

N.B. [Purple texts are linked to online resources]

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Since 1980s, many **new mathematical conjectures** have arisen

from the study of **string theory** and/or **supersymmetric QFTs**.

Two most famous examples are:

- **Mirror symmetry of Calabi-Yau manifolds**

- **Seiberg-Witten theory of four-dimensional manifolds**

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Two most famous examples are:

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- **Seiberg-Witten theory of four-dimensional manifolds**
  $\Longleftarrow$ came from **4d supersymmetric QFT**

Today I discuss a new **mathematical conjecture**
which we recently found from the properties of **4d susy QFTs**.

It seems much, much shallower than
mirror symmetry or Seiberg-Witten theory.

But it gives a **simple case of how such a conjecture arises from physics.**

1. **Conjecture**

2. **Physics derivation**

3. **Mathematical proof of a sub-case**

1. Conjecture

Let me start with very general definitions.

Let $\mathbf{Hom}(\Gamma, G)$ be the set of homomorphisms $f : \Gamma \to G$.

Let $\mathbf{Rep}(\Gamma, G)$ be $\mathbf{Hom}(\Gamma, G)/\sim$, where

$$f \sim f' \iff \exists g \in G \quad \text{s.t.} \quad g f(\gamma) g^{-1} = f'(\gamma) \text{ for all } \gamma \in \Gamma.$$

(When $G = U(N)$ this is just the set of isomorphism classes of N-dimensional complex representations of Γ, hence the notation.)

We then have two ingredients:

- Finite subgroups $\Gamma \subset SU(2)$, and
- Langlands dual pairs $(G, \widetilde{G})$ of compact simple Lie groups.

Let me discuss them in turn.

Let $\Gamma \subset SU(2)$ be a finite subgroup of $SU(2)$.

They are:

- $\mathbb{Z}_n \subset SU(2)$ generated by

$$\begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}$$

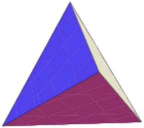
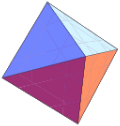
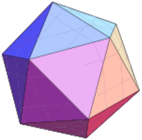
- $\widehat{D}_{4n} \subset SU(2)$ generated by

$$\begin{pmatrix} e^{2\pi i/(2n)} & 0 \\ 0 & e^{-2\pi i/(2n)} \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- And the binary tetrahedral, octahedral, icosahedral groups

$$\widehat{T}, \quad \widehat{O}, \quad \widehat{I}$$

which are the preimages under $SU(2) \to SO(3)$ of the symmetry groups of

 ,  ,  .

Famously they follow the A_n, D_n, $E_{6,7,8}$ classification:

$$\begin{aligned} A_{n-1} &\longleftrightarrow \mathbb{Z}_n, \\ D_{n+2} &\longleftrightarrow \widehat{D}_{4n}, \\ E_6 &\longleftrightarrow \widehat{T}, \\ E_7 &\longleftrightarrow \widehat{O}, \\ E_8 &\longleftrightarrow \widehat{I}. \end{aligned}$$

This is the McKay correspondence.

I'll come back to this later, if I have time.

Let G be a compact simple Lie group, and $\widetilde{G}$ be its Langlands dual.

They appear in **number theory** (in Langlands program)
and also in **4d SUSY QFT** (as part of S-duality).

The defining properties are:

- Dynkin diagrams for G and $\widetilde{G}$ have arrows reversed, and
- $\pi_1(G) = Z(\widetilde{G})$ and $Z(G) = \pi_1(\widetilde{G})$.

One example is to take

$$\mathfrak{g} = \widetilde{\mathfrak{g}} = \mathfrak{su}(n), \qquad \bullet - \bullet - \cdots - \bullet - \bullet$$

and then

$$G = SU(n), \qquad \widetilde{G} = SU(n)/\mathbb{Z}_n$$

so that

$$Z(G) = \pi_1(\widetilde{G}) = \mathbb{Z}_n, \qquad \pi_1(G) = Z(\widetilde{G}) = 1.$$

Another example is to take

$$\mathfrak{g} = \mathfrak{so}(2n+1), \qquad \bullet - \bullet - \cdots - \bullet \Rightarrow \bullet$$
$$\widetilde{\mathfrak{g}} = \mathfrak{sp}(n), \qquad \bullet - \bullet - \cdots - \bullet \Leftarrow \bullet$$

and

$$G = SO(2n+1), \qquad \widetilde{G} = Sp(n)$$

for which

$$Z(G) = \pi_1(\widetilde{G}) = 1, \qquad \pi_1(G) = Z(\widetilde{G}) = \mathbb{Z}_2$$

or

$$G = Spin(2n+1), \qquad \widetilde{G} = Sp(n)/\mathbb{Z}_2$$

for which

$$Z(G) = \pi_1(\widetilde{G}) = \mathbb{Z}_2, \qquad \pi_1(G) = Z(\widetilde{G}) = 1.$$

So, examples of pairs $(G, \widetilde{G})$ include:

$$(SU(n), SU(n)/\mathbb{Z}_n),$$

$$(SO(2n+1), Sp(n)),$$

$$(Spin(2n+1), Sp(n)/\mathbb{Z}_2).$$

Now we have introduced all the protagonists...

Conjecture

For a finite subgroup $\Gamma \subset SU(2)$ and
a Langlands dual pair $(G, \widetilde{G})$ of compact simple Lie groups,
we have

$$\left|\mathbf{Rep}(\Gamma, G)\right| = \left|\mathbf{Rep}(\Gamma, \widetilde{G})\right|$$

where $\mathbf{Rep}(\Gamma, G)$ is the set of
homomorphisms from Γ to G up to conjugation by G.

Remark

I do not expect there is a natural 1:1 map
between $\mathbf{Rep}(\Gamma, G)$ and $\mathbf{Rep}(\Gamma, \widetilde{G})$.

Rather, there are **two vector spaces**,

$$V(\Gamma, G) \quad \text{with bases labeled by} \quad \mathbf{Rep}(\Gamma, G)$$

and

$$V(\Gamma, \widetilde{G}) \quad \text{with bases labeled by} \quad \mathbf{Rep}(\Gamma, \widetilde{G})$$

such that there is a **nontrivial but natural isomorphism**

$$V(\Gamma, G) \simeq V(\Gamma, \widetilde{G}),$$

as we will see below.

Mathematical status of the conjecture

- $\Gamma = \mathbb{Z}_n$, $(G, \widetilde{G})$ arbitrary:

  already proved in [Djoković 1985]

- Γ arbitrary, $(G, \widetilde{G}) = (SU(n), SU(n)/\mathbb{Z}_n)$:

  we proved it in [arXiv:2505.01253],
  using McKay correspondence in a nice conceptual way

- Γ arbitrary, $(G, \widetilde{G}) = (SO(2n+1), Sp(n))$:

  we proved it in [arXiv:2505.01253],
  again using McKay correspondence, but less conceptually

Related works

$\mathbf{Rep}(\boldsymbol{Alt_5}, \boldsymbol{E_8})$ and $\mathbf{Rep}(\boldsymbol{SL(2, \mathbb{Z}_5)}, \boldsymbol{E_8})$ were studied by Frey in [Mem. AMS vol. 133, 1998].

Note that $\boldsymbol{Alt_5} \subset \boldsymbol{SO(3)}$ is the icosahedral group $\boldsymbol{I}$ and $\boldsymbol{SL(2, \mathbb{Z}_5)} \subset \boldsymbol{SU(2)}$ is its double cover $\widehat{\boldsymbol{I}}$.

This work caught the eyes of two famous mathematicians,
J. P. Serre and then G. Lustig,
who determined $\mathbf{Rep}(\boldsymbol{Alt_5}, \boldsymbol{G})$ but not $\boldsymbol{SL(2, \mathbb{Z}_5)}$
[Letter from Serre to Frey, 1998] [Lustig, J. Alg. **260** (2003) 298]

The fact that famous mathematicians got interested
does not imply the importance of the problem, but anyway ...

Plea to mathematicians in the audience

Please come up with a uniform proof of the conjecture!

It might not be deep, but it's concrete, and it might be fun.

Even proving some other subcases would be welcomed.

For example,

$$\Gamma = \text{ binary dihedral}, \qquad (G, \widetilde{G}) : \text{ arbitrary}$$

shouldn't be too hard...

2. Physics derivation

Recall the Maxwell equation in 4d:

$$d\,(\frac{1}{e^2}*F) = j_e,$$
$$dF = j_m$$

where

- F is the curvature of a $U(1)$ connection,
- $*$ is the Hodge star,
- e is the electric coupling, and
- j_e and j_m are the electric and magnetic currents.

Setting $j_e = j_m = 0$ for simplicity, we have

$$d\,(\frac{1}{e^2}{*F}) = 0,$$
$$dF = 0.$$

This has the symmetry

$$\frac{1}{e^2}{*F} \longleftrightarrow F.$$

In other words, by defining

$$\widetilde{F} := \frac{1}{e^2} * F,$$
$$\frac{1}{\widetilde{e}^{\,2}} * \widetilde{F} := F$$

where

$$e\widetilde{e} = 1,$$

we have

$$d\,(\frac{1}{\widetilde{e}^{\,2}} * \widetilde{F}) = 0,$$
$$d\widetilde{F} = 0$$

This is the **electromagnetic duality** of the Maxwell theory.
Known to hold quantum mechanically.

Maxwell theory is based on $U(1)$ connections.

Yang-Mills theory is obtained by replacing $U(1)$ with non-Abelian group G.

The equations are:

$$D(\frac{1}{e^2} * F) = 0,$$
$$DF = 0$$

where e^2 is the Yang-Mills coupling.

Is there an electromagnetic duality for Yang-Mills?

Well, it was found in [Goddard-Nuyts-Olive 1977] and [Montonen-Olive 1977] that:

- W-bosons (**electric** excitations) have charges controlled by **root vectors of $\mathfrak{g}$**,
- Monopoles (**magnetic** excitations) have charges controlled by **weight vectors of $\mathfrak{g}$**.

As

$$\text{weight lattice of } \mathfrak{g} = \text{root lattice of } \widetilde{\mathfrak{g}}$$

and vice versa,

the electromagnetic dual of $\mathfrak{g}$ Yang-Mills, if present, should be a $\widetilde{\mathfrak{g}}$ Yang-Mills.

But it soon became apparent that **it didn't work in vanilla Yang-Mills**.

It was soon realized that, if it ever had a chance to work,
we needed to consider $\mathcal{N}=4$ supersymmetric Yang-Mills.
[Olive-Witten 1978] [Osborn 1979]

(Witten himself did not believe that it would actually work at that time, as recounted in [this interview with Witten, 2014].)

Things changed thanks to [Sen hep-th/9402032] and many others.

Now **there are tons of evidence that it does work**.

So,

$$\mathcal{N}{=}\mathbf{4} \text{ super } \mathfrak{g} \text{ Yang-Mills at coupling } e \simeq$$

$$\mathcal{N}{=}\mathbf{4} \text{ super } \widetilde{\mathfrak{g}} \text{ Yang-Mills at coupling } \widetilde{e} = 1/e$$

How about the choice of the groups G and $\widetilde{G}$,
not just Lie algebras $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$?

G Yang-Mills theory is now known to have

electric one-form symmetry $Z(G)$

and

magnetic one-form symmetry $\pi_1(G)$

[Aharony-Seiberg-YT 1305.0318]
[Gaiotto-Kapustin-Seiberg-Willett 1412.5148]

The electromagnetic duality should then effect not only

Dynkin diagrams of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ have arrows reversed

but also

$$Z(G) = \pi_1(\widetilde{G}), \qquad \pi_1(G) = Z(\widetilde{G}).$$

That's exactly **the condition for a Langlands dual pair**.

So our current understanding is

$$\mathcal{N}{=}4 \text{ super } G \text{ Yang-Mills at coupling } e \simeq$$

$$\mathcal{N}{=}4 \text{ super } \widetilde{G} \text{ Yang-Mills at coupling } \widetilde{e} = 1/e$$

Known as S-duality.

The most basic approach to QFT is by a Taylor expansion in e or $\widetilde{e}$.
This duality is very difficult to see in such a basic approach.

Let $X(G, e)$ be a quantity X associated to $\mathcal{N}=4$ super G Yang-Mills at coupling e. Then we should have

$$X(G, e) = X(\widetilde{G}, \widetilde{e}), \qquad \text{where } \widetilde{e} = 1/e.$$

Typically we can only get the first few terms in the Taylor expansion in e or $\widetilde{e}$. Then checking this equality is hopeless.

But things change if X is a **quantity protected by supersymmetry**,
so that it is **independent** of e or $\widetilde{e}$.

In that case, $X(G, e) = X(G, 0)$ is computable,
and similarly $X(\widetilde{G}, \widetilde{e}) = X(\widetilde{G}, 0)$ is also computable.

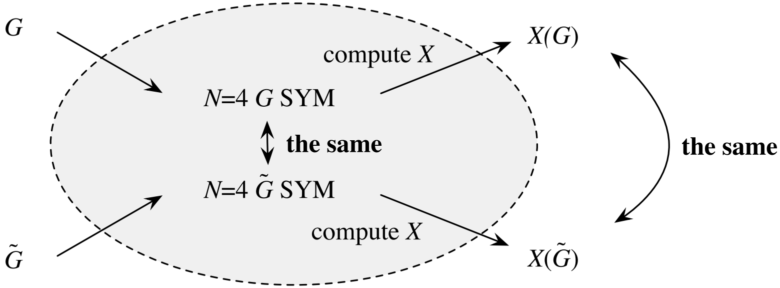
We then should have the equality

$$X(G, 0) = X(\widetilde{G}, 0).$$

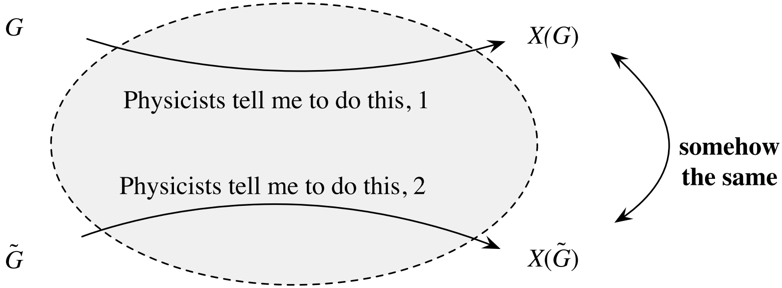
Note that this equality was 'derived' using S-duality,
a very nontrivial identity of supersymmetric QFTs.

But the final expressions for $X(G, 0)$ and $X(\widetilde{G}, 0)$
do not usually involve QFTs.

Then the equality becomes a well-defined conjecture in mathematics.



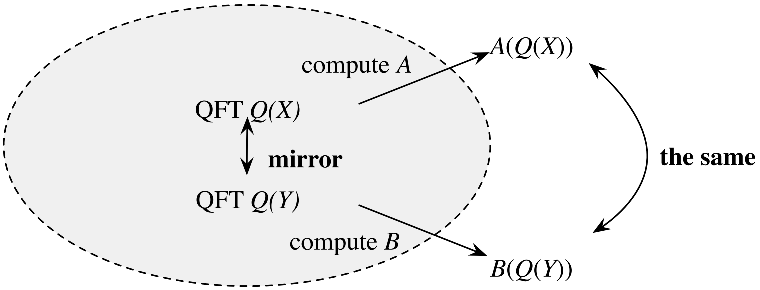
The shaded area is not yet well defined.



To mathematicians it becomes a conjecture.

Many other conjectures derived from supersymmetric QFTs follow the same pattern.

For example, homological mirror symmetry is



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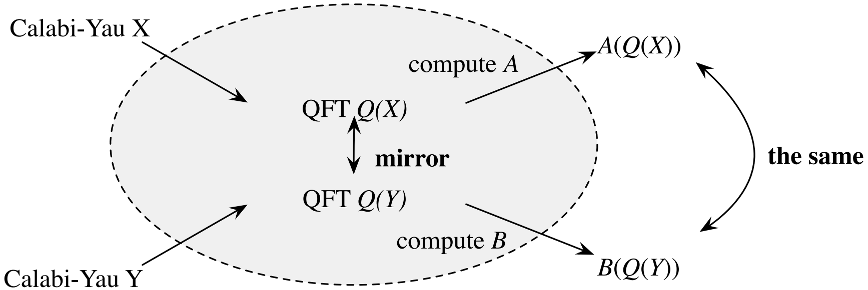

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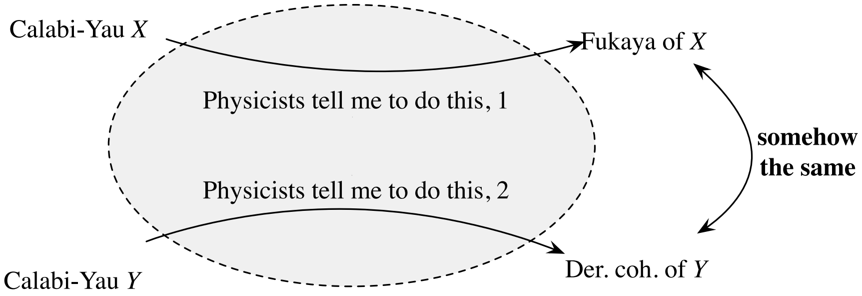


To proceed, we need to pick the protected quantity X.

For today's purpose, put $\mathcal{N}=4$ super G Yang-Mills on

$$S^3/\Gamma, \qquad \Gamma \subset SU(2).$$

Let $\mathcal{H}(\Gamma, G)$ be its Hilbert space of states, and
$V(\Gamma, G) \subset \mathcal{H}(\Gamma, G)$ be **the subspace of lowest-energy states**.
Then we will be interested in

$$X(\Gamma, G) := \mathrm{tr}_{V(\Gamma,G)}(-1)^F,$$

where $(-1)^F$ is a $\mathbb{Z}_2$ grading coming from fermion parity.

(For physicists: it is the coefficient of q^0 term
of the superconformal index on the lens space S^3/Γ.)

We compute $X(\Gamma, G)$ in the limit where the coupling constant e is zero.

For this, we need to identify **lowest-energy states**.

A configuration with nonzero field strength $F \neq 0$ has classical energy $\geq \int |F|^2$.

So a **necessary condition** is $F = 0$, i.e. **flat G bundles on S^3/Γ**, labeled by homomorphisms $f : \Gamma \to G$ up to conjugation by G.

This is **not a sufficient condition**.

Quantum fluctuations can give rise to non-zero zero-point energy known as **supersymmetric Casimir energy**

$$E(f : \Gamma \to G) \geq 0$$

in this context.

This is **generally non-zero in less supersymmetric theories**.

But with $\mathcal{N}=4$ supersymmetry, there is a huge cancellation between bosonic and fermionic contributions, so

$$E(f : \Gamma \to G) = 0$$

for any $f : \Gamma \to G$.

This was originally found to be true for some (Γ, G)
in [Ju, 2304.11830, 2311.18223].
We give a uniform derivation in [Choi-YT, to appear].

Each $f : \Gamma \to G$ (up to conjugation by G) contributes by $+1$ to $X(\Gamma, G)$, so

$$X(\Gamma, G) = \Big|\mathrm{Rep}(\Gamma, G)\Big|.$$

As $X(\Gamma, G) = X(\Gamma, \widetilde{G})$, we conclude

$$\Big|\mathrm{Rep}(\Gamma, G)\Big| = \Big|\mathrm{Rep}(\Gamma, \widetilde{G})\Big|.$$

So that's the physics 'derivation' of the conjecture.

3. Mathematical proof of a subcase

Let me now explain the proof of a sub-case of this conjecture:

$$\boldsymbol{\Gamma} \text{ arbitrary, } (\boldsymbol{G}, \widetilde{\boldsymbol{G}}) = (\boldsymbol{SU(n)}, \boldsymbol{SU(n)}/\mathbb{Z}_{\boldsymbol{n}}).$$

In addition,

- the sub-case $\boldsymbol{\Gamma} = \mathbb{Z}_{\boldsymbol{n}}$ and $(\boldsymbol{G}, \widetilde{\boldsymbol{G}})$ was proved in [Djoković 1985],
- and we also proved some other cases in [Kojima-YT 2505.01253].

I don't have the time to talk about them today.

We'd like to study

$$f : \Gamma \to SU(n)$$

and

$$f : \Gamma \to SU(n)/\mathbb{Z}_n.$$

Let's relate them to

$$f : \Gamma \to U(n).$$

Recall the map $\mathbf{det} : U(n) \to U(1)$.

$f : \Gamma \to SU(n)$ are those $f : \Gamma \to U(n)$ such that

$$f : \Gamma \to U(n) \xrightarrow{\mathbf{det}} U(1)$$

is trivial.

Let's rephrase it.

Let C be the group of one-dimensional representations,

$$C := \{f : \Gamma \to U(1)\},$$

where the group operation is the tensor product.

Recall that $\mathbf{Rep}(\Gamma, G)$ was

$$f : \Gamma \to G$$

up to conjugation in G.

Then $\mathbf{det}$ gives a map from $\mathbf{Rep}(\Gamma, U(n))$ to C, and

$$\mathbf{Rep}(\Gamma, SU(n)) = \mathbf{Ker}(\mathbf{det} : \mathbf{Rep}(\Gamma, U(n)) \to C).$$

Next consider

$$f : \Gamma \to SU(n)/\mathbb{Z}_n.$$

This means that f is a **projective** representation of Γ.

It turns out that $H^2(B\Gamma, U(1)) = 0$ for $\Gamma \subset SU(2)$,
so any projective representation lifts to a genuine representation

$$\widetilde{f} : \Gamma \to U(n).$$

Two such genuine representations

$$\widetilde{f}, \widetilde{f}' : \Gamma \to U(n)$$

descend to the same projective representation

$$f : \Gamma \to SU(n)/\mathbb{Z}_n$$

if and only if there is a homomorphism $c : \Gamma \to U(1)$ such that

$$\widetilde{f}' = c \otimes \widetilde{f}$$

under the tensor product, or equivalently the pointwise product

$$\widetilde{f}'(\gamma) = c(\gamma)\widetilde{f}(\gamma), \qquad \text{for all } \gamma \in \Gamma.$$

In other words, there is an action of

$$\boldsymbol{C = \{f : \Gamma \to U(1)\}}$$

on $\mathbf{Rep}(\boldsymbol{\Gamma}, \boldsymbol{U(n)})$, and

$$\mathbf{Rep}(\boldsymbol{\Gamma}, \boldsymbol{SU(n)/\mathbb{Z}_n}) = \mathbf{Rep}(\boldsymbol{\Gamma}, \boldsymbol{U(n)})/\boldsymbol{C}.$$

So:

$$\begin{aligned}\mathbf{Rep}(\Gamma, SU(n)) &= \mathbf{Ker}(\mathbf{det} : \mathbf{Rep}(\Gamma, U(n)) \to C),\\ \mathbf{Rep}(\Gamma, SU(n)/\mathbb{Z}_n) &= \mathbf{Rep}(\Gamma, U(n))/C,\end{aligned}$$

and we want to show

$$\left|\mathbf{Rep}(\Gamma, SU(n))\right| = \left|\mathbf{Rep}(\Gamma, SU(n)/\mathbb{Z}_n)\right|.$$

So we now first need to understand

$$\mathbf{Rep}(\boldsymbol{\Gamma}, \boldsymbol{U(n)}).$$

This is just the set of isomorphism classes of $\boldsymbol{n}$-dimensional complex representations of $\boldsymbol{\Gamma}$.

So, denoting by $\boldsymbol{\rho_i}$ the $\boldsymbol{i}$-th irreducible complex representation of $\boldsymbol{\Gamma}$, this set is in 1-to-1 correspondence with the solutions to

$$\boldsymbol{n = \sum_i n_i \dim \rho_i}.$$

We need to understand $\boldsymbol{\rho_i}$.

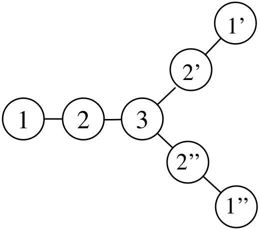
McKay correspondence

For $\Gamma \subset SU(2)$, do the following:

- Let V be the standard two-dimensional representation coming from $\Gamma \subset SU(2)$.
- Draw a vertex for each ρ_i.
- Draw an edge between ρ_i and ρ_j if and only if $V \otimes \rho_i$ contains ρ_j as an irreducible component.

You get an extended Dynkin diagram of type ADE.

For example, for $\boldsymbol{\Gamma = \widehat{T}}$, one gets:



the $\boldsymbol{E_6}$ extended Dynkin diagram!

For $\Gamma \subset SU(2)$, denote its ADE type by $\mathfrak{g}$.

Note that **this is not** the Lie algebra of G of (Γ, G). The solutions to

$$n = \sum_i n_i \dim \rho_i$$

are known to be the 1-to-1 correspondence to

$$\mathbf{Rep}(\widehat{\mathfrak{g}}, n) := \left\{ \begin{array}{c} \text{irreducible integrable representations of} \\ \text{affine } \widehat{\mathfrak{g}} \text{ Lie algebra at level } n \end{array} \right\}.$$

So,

$$\mathbf{Rep}(\Gamma, U(n)) \simeq \mathbf{Rep}(\widehat{\mathfrak{g}}, n).$$

We want to study

$$\mathbf{Rep}(\Gamma, SU(n)) = \mathbf{Ker}(\mathbf{det} : \mathbf{Rep}(\Gamma, U(n)) \to C)$$

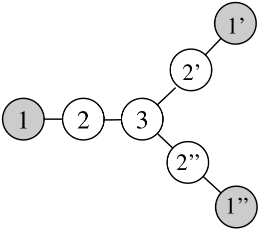
and

$$\mathbf{Rep}(\Gamma, SU(n)/\mathbb{Z}_n) = \mathbf{Rep}(\Gamma, U(n))/C.$$

What's this C on the RHS?

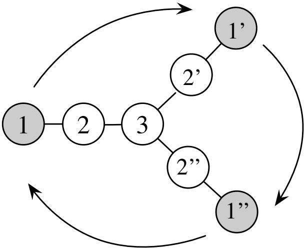
C was the group of 1-dimensional representations of Γ.

For $\Gamma = \widehat{T}$, for which $\mathfrak{g} = \mathfrak{e}_6$,



shows that $C = \mathbb{Z}_3$.

This $\boldsymbol{C}$ generates the graph automorphism



which becomes outer automorphism group of $\widehat{\boldsymbol{g}}$.

As such, $\boldsymbol{C}$ naturally acts on

$$\mathbf{Rep}(\widehat{\mathfrak{g}}, n).$$

Denoting by $A^\vee$ the Pontryagin dual of the finite Abelian group A, it is also known that $C^\vee$ agrees with

$$Z(\text{simply-connected Lie group of type } \mathfrak{g}).$$

For example $Z(E_6) = \mathbb{Z}_3$.

(As abstract groups they are the same, $A^\vee \simeq A$, but not canonically.)

Therefore $C^\vee$ acts as a scalar multiplication in an irreducible integrable representation of $\widehat{g}$, determining a map

$$\mathbf{Rep}(\widehat{\mathfrak{g}}, n) \to C.$$

The map $\mathbf{det}: \mathbf{Rep}(\Gamma, U(n)) \to C$ agrees with this

$$\mathbf{Rep}(\widehat{\mathfrak{g}}, n) \to C.$$

Similarly, the C action on $\mathbf{Rep}(\Gamma, U(n))$ by tensor product agrees with the C action on

$$\mathbf{Rep}(\widehat{\mathfrak{g}}, n)$$

by outer automorphism of $\widehat{\mathfrak{g}}$.

Let $V(\widehat{\mathfrak{g}}, n)$ be the vector space with basis v_a labeled by

$$a \in \mathbf{Rep}(\widehat{\mathfrak{g}}, n).$$

This is the vector space of **affine characters of $\widehat{\mathfrak{g}}$ at level n**.

This is also the Hilbert space of $\mathfrak{g}$ **Chern-Simons theory at level n on T^2**.

Now, the C-grading

$$\mathbf{Rep}(\widehat{\mathfrak{g}}, n) \to C$$

gives an action of $C^\vee$ on $V(\widehat{\mathfrak{g}}, n)$.

Similarly, the C action on $\mathbf{Rep}(\widehat{\mathfrak{g}}, n)$ gives a C action on $V(\widehat{\mathfrak{g}}, n)$.

Then

$$\left|\mathbf{Ker}(\mathbf{Rep}(\widehat{\mathfrak{g}}, n) \to C)\right| = \dim V(\widehat{\mathfrak{g}}, n)^{C^\vee}$$

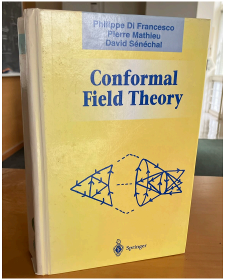
and

$$\left|\mathbf{Rep}(\widehat{\mathfrak{g}}, n)/C\right| = \dim V(\widehat{\mathfrak{g}}, n)^{C}.$$

Now, the C action and the $C^\vee$ action on $\dim V(\widehat{\mathfrak{g}}, n)$ have been studied exhaustively in the 1990s by 2d CFT theorists, and are known to be conjugate by the action of

$$S \in SL(2,\mathbb{Z}) \curvearrowright V(\widehat{\mathfrak{g}}, n).$$

See e.g. Sec. 14.6 of [Di Francesco-Mathieu-Sénéchal]:



So

$$\dim V(\widehat{\mathfrak{g}}, n)^{C^\vee} = \dim V(\widehat{\mathfrak{g}}, n)^C,$$

meaning

$$\left|\mathrm{Ker}(\mathrm{Rep}(\widehat{\mathfrak{g}}, n) \to C)\right| = \left|\mathrm{Rep}(\widehat{\mathfrak{g}}, n)/C\right|,$$

meaning

$$\left|\mathrm{Ker}(\mathrm{Rep}(\Gamma, U(n)) \to C)\right| = \left|\mathrm{Rep}(\Gamma, U(n))/C\right|,$$

meaning

$$\left|\mathrm{Rep}(\Gamma, SU(n))\right| = \left|\mathrm{Rep}(\Gamma, SU(n)/\mathbb{Z}_n)\right|,$$

proving this subcase of the conjecture.

Summary

- A mathematical conjecture.
- Its physics derivation.
- Proofs of some subcases.

Questions?

Proof of another sub-case

In this appendix we discuss another sub-case of the conjecture,

$$\Gamma = \mathbb{Z}_n \text{ and } (G, \widetilde{G}) \text{ arbitrary.}$$

This was proved in [Djoković 1985], as already mentioned.

Here we review the proof for illustration.
It 'feels' very similar to the other proof, as you will see.

A homomorphism $f : \mathbb{Z}_n \to G$ is specified by $f(1) \in G$ satisfying $f(1)^n = e$.

There is a Cartan torus $T \subset G$ such that $f(1) \in T$.

Write $T = \mathfrak{t}/\Lambda$.

$f(1)^n = 1$ means that $f(1)$ defines an element of $A := \Lambda/n\Lambda$. So

$$\mathbf{Rep}(\mathbb{Z}_n, G) \simeq A/W$$

where W is the Weyl group.

We now introduce a complex vector space $V(A)$ with basis v_a, $a \in A$. Clearly

$$\Big|\,\mathbf{Rep}(\mathbb{Z}_n, G)\Big| = \Big|A/W\Big| = \dim V(A)^W$$

where $V(G)^W$ is the W-invariant subspace.

Performing the same analysis on the $\widetilde{G}$ side, we have

$$\left|\mathbf{Rep}(\mathbb{Z}_n, G)\right| = \dim V(A)^W,$$
$$\left|\mathbf{Rep}(\mathbb{Z}_n, \widetilde{G})\right| = \dim V(\widetilde{A})^W,$$

where

$$A := \Lambda/n\Lambda, \qquad \widetilde{A} := \widetilde{\Lambda}/n\widetilde{\Lambda}.$$

Recall that $V(A)$ has the basis v_a and $V(\widetilde{A})$ has the basis $\widetilde{v}_{\widetilde{a}}$.

We can show that A and $\widetilde{A}$ are naturally Pontryagin dual to each other, and the matrix

$$\Big(\langle a, \widetilde{a} \rangle\Big)_{a,\widetilde{a}}$$

is non-degenerate and W invariant.

This allows us to introduce a non-degenerate W-invariant pairing between $V(A)$ and $V(\widetilde{A})$ by

$$(v_a, \widetilde{v}_{\widetilde{a}}) := \langle a, \widetilde{a} \rangle.$$

Then the irreducible decomposition of $V(A)$ and $V(\widetilde{A})$ under the W action is the same, implying in particular

$$\dim V(A)^W = \dim V(\widetilde{A})^W,$$

proving this subcase of the conjecture.