On a rarely-mentioned class of 3d $\mathcal{N}=4$ SCFTs

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Osaka City U., March 24, 2022

(Phrases in purple are hyperlinked if you download the slides.)

When Jaewon and Kazunobu invited me to this workshop in December,
I accepted to give a talk, without thinking too much.
But later I realized that this is mostly a SUSY QFT workshop,
and that I don't have any suitable recent work to talk about.

So I decided to revive in February an unpublished long-dormant project
based on an idea I had long time ago (circa 2015?).
Since its inception I collaborated with many persons on this project:
Jenny Wong, Benjamin Assel,
Alessandro Tomasiello, Seyed Morteza Hosseini.

Luckily we made some progress since last month that I can report on.

I have **two** introductions from distinct points of view:

- One from highly-supersymmetric Chern-Simons-matter theories
- Another from M5-branes on 3-manifolds

They somehow flow to the same endpoint...

Highly-supersymmetric Chern-Simons theories

Supersymmetric Chern-Simons terms are available up to $\mathcal{N}=\mathbf{3}$.

Realizing $\mathcal{N} \geq \mathbf{4}$ was thought impossible.

Breakthrough in Autumn 2007 – Early summer 2008:

- $\mathcal{N}=\mathbf{8}$ [Gustavsson 0709.1260]
  [Bagger-Lambert 0711.0955]
- $\mathcal{N}=\mathbf{4}$ [Gaiotto-Witten 0804.2907]
  [Hosomichi-Lee-Lee-Lee-Park 0805.3662]
- $\mathcal{N}=\mathbf{6}$ [Aharony-Bergman-Jafferis-Maldacena 0806.1218]
- $\mathcal{N}=\mathbf{5}$ [Hosomichi-Lee-Lee-Lee-Park 0806.4977]

They used various approaches,
which in the end can be explained uniformly as follows:

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- $\mathcal{N}=5$ [Hosomichi-Lee-Lee-Lee-Park 0806.4977]

Contained ABJM as a special case. If only they realized $\mathcal{N}=6$ enhancement...

They used various approaches,
which in the end can be explained uniformly as follows:

Mechanism of the enhancement

Let us start with an exercise in 4d.

We can achieve $\mathcal{N}{=}2$ with general G and R.

When R is the adjoint rep., the superpotential in $\mathcal{N}{=}1$ language is

$$W = \mathrm{tr}\,\Phi[A, B]$$

where Φ is from the vector multiplet and A, B form a hypermultiplet.

This has $SU(3)$ acting on Φ, A, B.

This **does not** commute with $SU(2)_R$ of $\mathcal{N}{=}2$ SUSY.

$\Rightarrow$ SUSY has to enhance, giving $\mathcal{N}{=}4$ SYM.

In 3d, we can achieve $\mathcal{N}=3$ with general G and R.
The superpotential in $\mathcal{N}=2$ language is

$$W = \sum_i (\operatorname{tr} \Phi_i \mu_i - \frac{k_i}{2} \operatorname{tr} \Phi_i^2)$$

where Φ_i is the adjoint scalar in the Chern-Simons supermultiplet and μ_i is the **moment map operator**, constructed from the hypers.

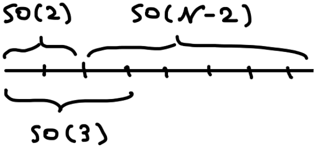
As Φ_i's have no kinetic term, they can be integrated out, giving

$$W \propto \sum_i \frac{1}{k_i} \operatorname{tr} \mu_i^2.$$

When k_i is chosen carefully, W can have
$\mathcal{N}=2$ flavor symmetry **not commuting** with $\mathcal{N}=3$ R-symmetry.

$\rightarrow$ **SUSY enhancement!**

If the final result has $\mathcal{N}=N$ SUSY,
we expect $SO(N-2)$ flavor symmetry in the $\mathcal{N}=2$ formalism:



For example, in the ABJM case,
we expect to see $SO(6-2) = SO(4)$ emerging.

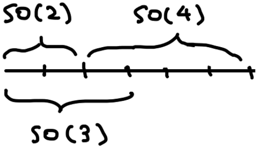
Indeed, $U(N)_k \times U(N')_{-k'}$ with two bifundamentals A_i, B^i has

$$W \propto \frac{1}{k}\,\mathrm{tr}(A_i B^i)^2 - \frac{1}{k'}\,\mathrm{tr}(B^i A_i)^2$$

which simplifies, if $k = k'$, to

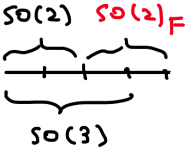
$$W \propto \frac{1}{k}\,\mathrm{tr}\,A_i B^a A_j B^b \epsilon^{ij} \epsilon_{ab}.$$

So we indeed see $SU(2) \times SU(2) = SO(6-2)$.



(I will be sloppy about the global structure of groups in this talk.)

Enhancement to $\mathcal{N}=4$ is in a sense simpler.



We couple $\mathcal{N}=3$ Chern-Simons term to $\mathcal{N}=4$ matter system.

In $\mathcal{N}=2$ language, $\mathcal{N}=4$ matter system has $SO(2)_F$ flavor symmetry, under which the moment map operator μ has charge $+1$, say.
This is broken by the superpartner of the Chern-Simons term:

$$W = \sum_i (\operatorname{tr}\Phi_i \mu_i - \frac{k_i}{2} \operatorname{tr}\Phi_i^2).$$

After integrating out Φ_i, we have

$$W \propto \underbrace{\sum_i \frac{1}{k_i} \operatorname{tr} \mu_i^2}_{\text{charge } +2} .$$

The $SO(2)_F$ symmetry assigning charge $+1$ to μ_i is broken generically, but it can happen that $W \propto \sum_i \frac{1}{k_i} \operatorname{tr} \mu_i^2 = 0$.

Then we see the enhancement back to $\mathcal{N}=4$.

The observation of Gaiotto and Witten was that,
for G with a half-hyper in R,

$$W \propto \sum_i \frac{1}{k_i} \operatorname{tr} \mu_i^2 = 0$$

is equivalent to $G \oplus R$ forming a **super Lie algebra** $\mathcal{G}$.

- G is the **bosonic part** and R is the **fermionic part** of $\mathcal{G}$.
- The ratio $k_1 : k_2 : \cdots$ is part of the **structure constants** of $\mathcal{G}$.
- Rozansky-Witten twist of this $\mathcal{N}=4$ theory is the Chern-Simons theory of $\mathcal{G}$. [Kapustin-Saulina 0904.1447]

For example,

$U(N|M)$ gives
$U(N)_k \times U(M)_{-k}$ with **one** bifundamental hyper.

(With **two** bifundamentals, it enhances further to $\mathcal{N}=6$.)

$OSp(N|M)$ gives
$SO(N)_k \times Sp(M)_{-2k}$ with **one** bifundamental half-hyper.

(With **two** bifundamentals, it enhances further to $\mathcal{N}=5$.)

In both cases, the structure constant of the Lie algebras is unique
$\Leftrightarrow$ the ratio of Chern-Simons levels is fixed

The case $OSp(4|2)$ is special.

The gauge group is $SO(4) \times SU(2) = SU(2)_1 \times SU(2)_2 \times SU(2)_3$.

The half-hyper is in $\mathbf{4} \otimes \mathbf{2} = \mathbf{2_1} \otimes \mathbf{2_2} \otimes \mathbf{2_3}$.

Denote it by Q_{aiu} as always. The condition for the SUSY enhancement is

$$W \propto \frac{1}{k_1} \operatorname{tr}(\mu_1)^2 + \frac{1}{k_2} \operatorname{tr}(\mu_2)^2 + \frac{1}{k_3} \operatorname{tr}(\mu_3)^2 = 0$$

where $\mu_{1,2,3}$ are the moment map operators for $SU(2)_{1,2,3}$.

Three moment map operators are

$$\mu_1{}^c{}_b = \epsilon^{ca}\mu_{1,ab}\ , \qquad \mu_{1,ab} = \epsilon^{ij}\epsilon^{uv}Q_{aiu}Q_{bjv},$$
$$\mu_2{}^k{}_j = \epsilon^{ki}\mu_{2,ij}\ , \qquad \mu_{2,ij} = \epsilon^{ab}\epsilon^{uv}Q_{aiu}Q_{bjv},$$
$$\mu_3{}^w{}_v = \epsilon^{wv}\mu_{3,uv}\ , \qquad \mu_{3,uv} = \epsilon^{ab}\epsilon^{ij}Q_{aiu}Q_{bjv}$$

and it turns out

$$\mathrm{tr}(\mu_1)^2 = \mathrm{tr}(\mu_2)^2 = \mathrm{tr}(\mu_3)^2.$$

Therefore, the SUSY enhances to $\mathcal{N}=4$ if and only if

$$W \propto \frac{1}{k_1}\mathrm{tr}(\mu_1)^2 + \frac{1}{k_2}\mathrm{tr}(\mu_2)^2 + \frac{1}{k_3}\mathrm{tr}(\mu_3)^2 = 0$$

i.e. if and only if

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 0.$$

A weird condition!

As I said, the ratio $k_1 : k_2 : \cdots$ is part of the structure constants of the super Lie algebra.

The condition

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 0$$

means that there is a one-parameter family of the ratio.

Therefore, the super Lie algebra $OSp(4|2)$ comes in a one-parameter family.

In fact, it is the only simple super Lie algebra which comes in a continuous family.

Often denoted as $D(2, 1; \alpha)$.

For us, $k_{1,2,3}$ need to be integers. Are there integer solutions to

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 0 \quad ?$$

Yes,

$$\frac{1}{p(p+q)} + \frac{1}{q(p+q)} - \frac{1}{pq} = 0,$$

for example.

We can generalize. If you like class S theories, you know that the trifundamental Q_{aiu} of $SU(2)$ is the $T_{SU(2)}$ theory.

More generally, there is a 3d $\mathcal{N}=4$ theory known as the T_G theory with G^3 symmetry.

It is obtained by putting 6d (2,0) theory of type G on S^2 with three full punctures, further compactified on S^1.

It has moment map operators $\mu_{1,2,3}$ for G^3 symmetry.

They satisfy

$$\mathrm{tr}(\mu_1)^n = \mathrm{tr}(\mu_2)^n = \mathrm{tr}(\mu_3)^n$$

for arbitrary n.

Consider, then, the 3d $\mathcal{N}=4$ T_G theory coupled with $\mathcal{N}=3$ Chern-Simons terms $G_{k_1} \times G_{k_2} \times G_{k_3}$.

Then, the condition for the enhancement to $\mathcal{N}=4$ is

$$W \propto \frac{1}{k_1}\,\mathrm{tr}(\mu_1)^2 + \frac{1}{k_2}\,\mathrm{tr}(\mu_2)^2 + \frac{1}{k_3}\,\mathrm{tr}(\mu_3)^2 = 0$$

i.e.

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 0.$$

This is (the simplest type of) the rarely-mentioned class of 3d $\mathcal{N}=4$ theory in the title.

M5-branes on 3-manifolds

People love considering M5-branes on 3-manifolds, starting from [Terashima-Yamazaki 1103.5748] and [Dimofte-Gaiotto-Gukov 1108.4389]

What kind of 3d theories do we get?

Before that, how many supersymmetries do we have?

That's the question I'd like to raise today.

(N.B. In the first day of the workshop, [Dongmin Gang] told us also about $\mathcal{N}=4$ enhancement among these theories. There are some differences, though. He talked about $T_{\text{irred}}[M]$ while I will talk about $T_{\text{full}}[M]$, in his notation.)

M5-branes on T^3 gives 3d $\mathcal{N}=8$, obviously.

M5-branes on generic M_3 can preserve 3d $\mathcal{N}=2$.

Indeed, the holonomy of M_3 is in $SO(3)_M$.
The R-symmetry of the 6d theory is $SO(5)_R$.
We turn on the R-symmetry background by embedding $SO(3)_M$ to

$$SO(3) \times SO(2) \subset SO(5)_R$$

Then $SO(2)$ remains as a 3d R-symmetry, leading to 3d $\mathcal{N}=2$.

As another big class, M5-branes on $\Sigma_2 \times S^1$ gives 3d $\mathcal{N}=4$.

The holonomy of Σ_2 is in $SO(2)_\Sigma$.
The R-symmetry of the 6d theory is $SO(5)_R$.
We turn on the R-symmetry background by embedding $SO(2)_\Sigma$ to

$$SO(2) \times SO(3) \subset SO(5)_R.$$

Then $SO(2) \times SO(3)$ remains as a 3d R-symmetry.
(Note that $SO(2)$ commutes with itself!)

But in 3d $\mathcal{N}=N$ superconformal theories, the R-symmetry is $SO(N)_R$.
By carefully studying how the R-symmetry acts on the supercharges, we find that

$$SO(2) \times SO(3) \subset SO(3) \times SO(3) \simeq SO(4)_R,$$

i.e. we have 3d $\mathcal{N}=4$.
Note that we only used the fact that the holonomy is in $SO(2)$.

How about the other cases?

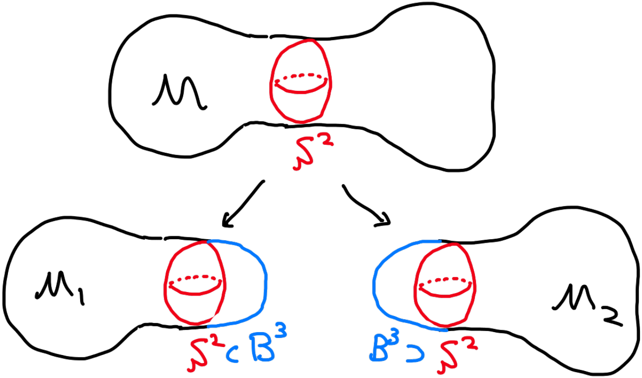
To study these points, it is useful to recall

the classification of 3-manifolds

(N.B. [Dongmin Gang] also gave a nice review on the first day.)

Prime decomposition

First step is to decompose 3-manifolds along shared S^2.

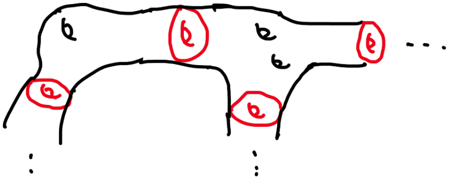


Such decomposition is known to be essentially unique:

$$M = M_1 \# M_2 \# \cdots \# M_n.$$

Torus decomposition

Second step is to decompose prime 3-manifolds along shared T^2.



In this step, as there are multiple ways to fill T^2 in,
we keep each piece to have (multiple) torus boundaries.

This decomposition is not quite unique
but all the non-uniqueness comes from well-understood examples.

A piece which cannot be further decomposed
by cutting along $\boldsymbol{T^2}$ is called **atoroidal**.

$$\textbf{Any 3-manifold} \xrightarrow{\text{cut along } \boldsymbol{S^2}} \textbf{prime 3-manifolds}$$

$$\textbf{Prime 3-manifold} \xrightarrow{\text{cut along } \boldsymbol{T^2}} \textbf{atoroidal 3-manifolds}$$

(Essential uniqueness of these decompositions was proved by 1979.)

What are atoroidal 3-manifolds, then?

Geometrization

Atoroidal manifolds are either

- Seifert fibrations, or
- hyperbolic, i.e. $\mathbb{H}^3/\Gamma$.

Originally conjectured by Thurston in 1982.

Proved by Perelman in 2006 using the Ricci flow.

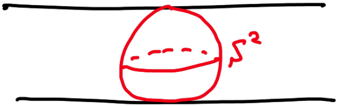
So, to understand M5-branes on 3-manifolds requires a few steps:

- Understand M5-branes on Seifert fibrations.
- Understand M5-branes on hyperbolic manifolds.

And then

- Understand what happens when glued along T^2.
- Understand what happens when glued along S^2.

What happens when glued along S^2

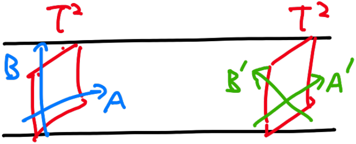


I haven't see any paper about it.

As M5-branes on S^2 give a gapped theory, nothing happens, presumably.

Some TQFT effects? I don't know.

What happens when glued along T^2



M5-branes on T^2 give $\mathcal{N}=4$ SYM with gauge group G.
Then, a torus boundary means a G flavor symmetry for a 3d theory.

Gluing the two $=$ gauging them with an $SL(2,\mathbb{Z})$ wall,
which can be decomposed to

$$T^{k_1} S T^{k_2} S \cdots T^{k_{n-1}} S T^{k_n},$$

where T^k: $\mathcal{N}=3$ Chern-Simons term and S: $\mathcal{N}=4$ $T[G]$ theory.

M5-branes on hyperbolic manifolds

Tons of papers!

Goes back to:

[Terashima-Yamazaki 1103.5748] for mapping tori

[Dimofte-Gaiotto-Gukov 1108.4389] for triangulations

Important refinements e.g. in:

[Chung-Dimofte-Gukov-Sułkowski 1405.3663]

[Gang-Yonekura 1803.04009]

This is not the place to review them.

M5-branes on Seifert fibrations

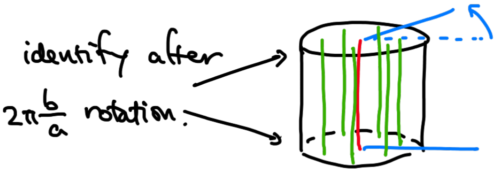
Basically determined in a series of works, e.g.

[Gadde-Gukov-Putrov 1306.4320]
[Pei-Ye 1503.04809]
[Gukov-Putrov-Vafa 1602.05302]
[Gukov-Pei-Putrov-Vafa 1701.06567]
[Eckhard, Kim, Schäfer-Nameki, Willett 1910.14086]
[Cho-Gang-Kim 2007.01532]

But I have a bit more to say today.

What is a Seifert fibration, anyway?

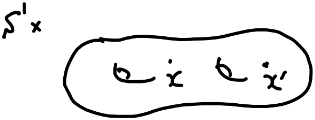
It is a slight generalization of an S^1 fibration over a surface Σ, where you allow **singular fibers** of the form



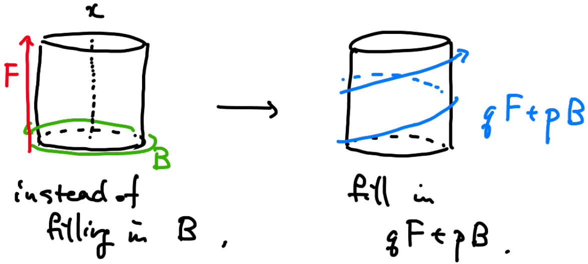
The 3d manifold is smooth, but the base Σ now has singularities of the form $\mathbb{C}/\mathbb{Z}_a$.

(a can vary from a singular fiber to another singular fiber.)

Another way to phrase the construction is: start from



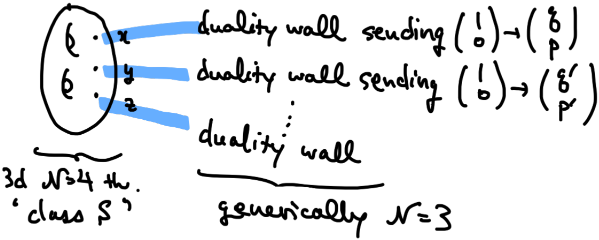
and perform



at a number of points. Seifert fibrations of type

$$\left(g; \frac{q_1}{p_1}, \frac{q_2}{p_2}, \ldots, \frac{q_n}{p_n}\right).$$

The corresponding 3d theory is therefore

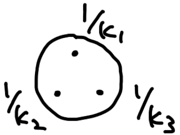


So we can make it $\mathcal{N}=3$, field theoretically.

I don't understand this enhancement from the geometric point of view, since the holonomy is generically $SO(3)$, which leads to $\mathcal{N}=2$.

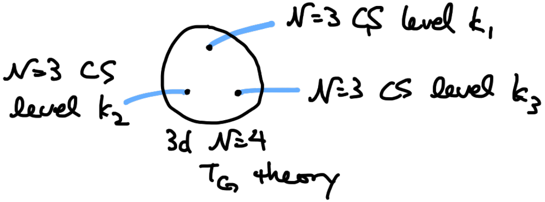
And the original paper [Gukov-Putrov-Vafa 1602.05302] didn't mention the enhancement.

More concretely, consider M5-branes on a Seifert of type



.

This leads to



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Now, let me point out something new.

The Seifert fibration of type

$$(g; \frac{q_1}{p_1}, \frac{q_2}{p_2}, \dots, \frac{q_n}{p_n})$$

is known to have $SO(2)$ holonomy when

$$\sum \frac{q_i}{p_i} = 0,$$

although Seifert fibrations generically have $SO(3)$ holonomy.

Why?

Well, given a Seifert fibration $M \to \Sigma$ of type $(g; \frac{q_1}{p_1}, \frac{q_2}{p_2}, \ldots, \frac{q_n}{p_n})$, we consider the following pull back:

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "\text{genuine } S^1 \text{ fibration}"'] & M \arrow[d, "\text{Seifert fibration}"] \\ \Sigma' \arrow[r, "\prod p_i \text{ cover}"] & \Sigma \end{tikzcd}$$

and the first Chern class of $M' \to \Sigma'$ is

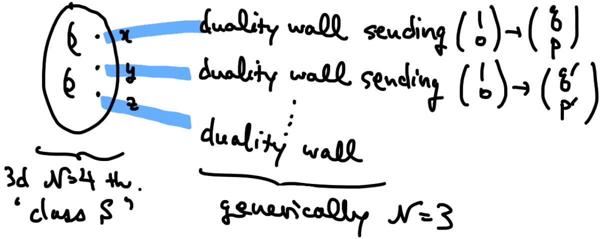
$$c_1(M') = (\prod p_i) \sum \frac{q_i}{p_i}.$$

So, M is a $\mathbb{Z}_{\prod p_i}$ shift-orbifold of $S^1 \times \Sigma'$ if

$$\sum \frac{q_i}{p_i} = 0.$$

Then, it is has $SO(2)$ holonomy.

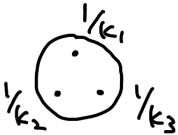
Therefore, the theories



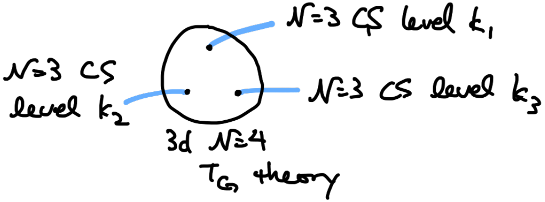
are $\mathcal{N}=4$ when

$$\sum \frac{q_i}{p_i} = 0.$$

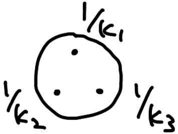
Let us come back to M5-branes on the Seifert of type



This led to



We just learned that the Seifert manifold

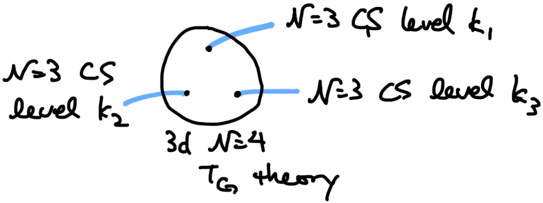


has $SO(2)$ holonomy iff

$$\sum \frac{1}{k_i} = 0,$$

leading to $\mathcal{N}=4$.

And we already saw in the field theory introduction that



has the superpotential

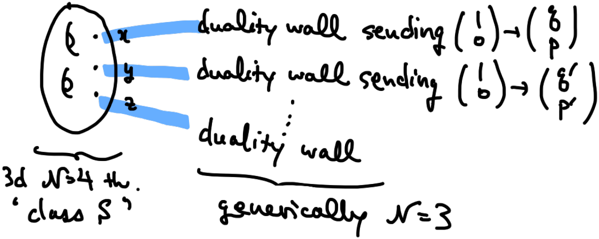
$$W \propto \sum \frac{1}{k_i} \operatorname{tr}(\mu_i)^2 = (\sum \frac{1}{k_i}) \operatorname{tr} \mu^2$$

which vanishes iff

$$\sum \frac{1}{k_i} = 0,$$

leading to $\mathcal{N}=4$. Neat, isn't it?

The field theory argument can be extended to all theories



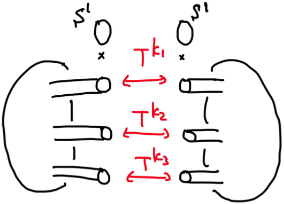
with

$$\sum \frac{q_i}{p_i} = 0$$

by using the properties of the S-duality wall theory $T[G]$. click for details

Geometrically Unexplained Enhancements

For example, consider

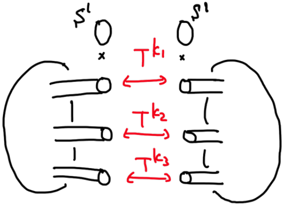


The superpotential is

$$W = \sum_i (\operatorname{tr} \mu_i \Phi_i - \frac{k_i}{2} \operatorname{tr} {\Phi_i}^2 + \operatorname{tr} \Phi_i \mu_i').$$

Integrating out Φ_i, we get

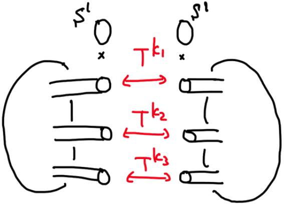
$$W \propto (\sum_i \frac{1}{k_i})(\operatorname{tr} \mu^2 + \operatorname{tr} \mu'^2) + \sum_i \frac{1}{k_i} \operatorname{tr} \mu_i \mu_i'.$$



When $\displaystyle\sum_i \frac{1}{k_i} = 0$, this theory ends up having $\displaystyle W = \sum_i \frac{1}{k_i} \operatorname{tr} \mu_i \mu_i'$.

We can now assign charge $+1$ to μ_i and charge -1 to μ_i', leading to $\mathcal{N}=4$ enhancement.

For two M5-branes, this reduces to a particular case of $\mathcal{N}=5$ theory of [Hosomichi-Lee-Lee-Lee-Park 0806.4977].



But geometrically, this is a mapping torus of a genus-2 surface over S^1, where we use Dehn twists around three necks to glue.

Nothing special happens when $\sum \frac{1}{k_i} = 0$, at far as I can see.

Summary

I considered how many supersymmetries are realized
when we put M5-branes on 3-manifolds without hyperbolic parts.

From geometry, $\mathcal{N}=2$ is expected generically,
with occasional enhancements to $\mathcal{N}=4$.

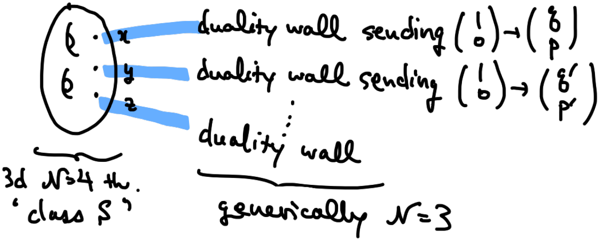
From field theory, $\mathcal{N}=3$ is expected generically,
with occasional enhancements to $\mathcal{N}=4$.

Sometimes they agree; sometimes they don't.

Back-up slides

More geometrically understandable cases

The field theory argument can be extended to all theories



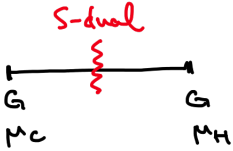
with

$$\sum \frac{q_i}{p_i} = 0$$

by using the properties of the S-duality wall theory $T[G]$.

(The notations $T[G]$ and T_G are confusing. Please blame Davide.)

The 3d $\mathcal{N}=4$ theory $T[G]$ implements the S-duality of 4d $\mathcal{N}=4$ SYM:



As such, it has $G \times G$ symmetry, one acting on the Coulomb branch and another acting on the Higgs branch.
Correspondingly, there are two moment map operators μ_C and μ_H.

We can perform $\mathcal{N}=4$-preserving mass deformations

$$W = \operatorname{tr} M_H \mu_C + \operatorname{tr} M_C \mu_H .$$

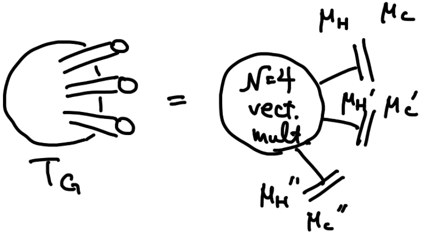
This is known to affect the operators so that

$$\operatorname{tr}(M_H)^n = \operatorname{tr}(\mu_H)^n, \qquad \operatorname{tr}(M_C)^n = \operatorname{tr}(\mu_C)^n .$$

This can be used to show the chiral ring relation

$$\mathbf{tr}(\mu_1)^n = \mathbf{tr}(\mu_2)^n = \mathbf{tr}(\mu_3)^n$$

of the 3d $\mathcal{N}=4$ T_G theory, because this theory has the description



of three $T[G]$ theories coupled together, with the superpotential

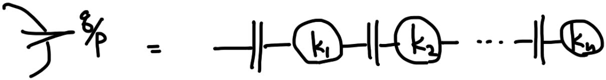
$$W = \sum_i \mathbf{tr}\,\Phi\mu_{i,H}$$

Then we have

$$\mathbf{tr}(\mu_{i,C})^n = \mathbf{tr}\,\Phi^n$$

independent of i.

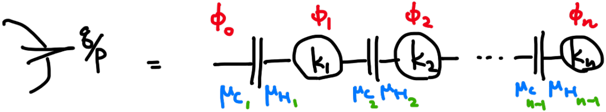
Now, a Seifert fiber of type q/p has a 3d $\mathcal{N}=3$ description



where $||$ is the $T[G]$ theory and k_i are connecting $\mathcal{N}=3$ CS terms, and the levels k_i are determined by writing q/p as a continued fraction:

$$\frac{q}{p} = \cfrac{1}{k_1 - \cfrac{1}{k_2 - \cfrac{1}{k_3 - \cdots}}}.$$

This part



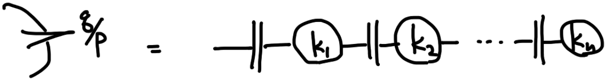
has the superpotential

$$
\begin{aligned}
W = {} & \operatorname{tr}\Phi_0(\mu_C)_1 + \operatorname{tr}(\mu_H)_1\Phi_1 - \frac{k_1}{2}\operatorname{tr}(\Phi_1)^2 \\
& + \operatorname{tr}\Phi_1(\mu_C)_2 + \operatorname{tr}(\mu_H)_2\Phi_2 - \frac{k_2}{2}\operatorname{tr}(\Phi_2)^2 \\
& + \cdots \\
& + \operatorname{tr}\Phi_{n-1}(\mu_C)_n + \operatorname{tr}(\mu_H)_n\Phi_n - \frac{k_n}{2}\operatorname{tr}(\Phi_n)^2.
\end{aligned}
$$

We now integrate out Φ_i, and use

$$\operatorname{tr}((\mu_C)_i)^2 = \operatorname{tr}(\Phi_i)^2 = \operatorname{tr}((\mu_H)_{i+1})^2.$$

At the end of the day, we find

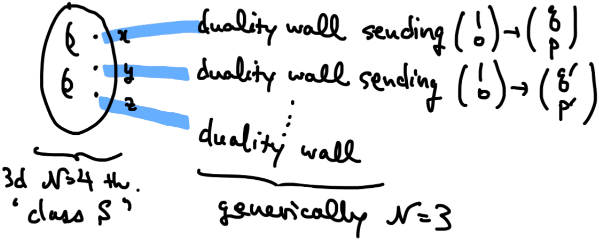


generates the superpotential

$$W \propto \frac{q}{p} \operatorname{tr}(\Phi_0)^2.$$

- I have done only a very crude field theoretical analysis; there are some gaps in the derivation.
- This is related to the fact that the (p, q) 5-brane is tilted by the factor p/q, and the observation goes back to [Kitao-Ohta-Ohta hep-th/9808111]. They noticed that this implied a 'fractional Chern-Simons level' of p/q and got confused.

So, the theory



ends up generating

$$W \propto \left(\sum \frac{q_i}{p_i}\right) \mathrm{tr}(\Phi_0)^2.$$

Therefore the theory enhances to $\mathcal{N}=4$ when

$$\sum \frac{q_i}{p_i} = 0.$$

go back