Dissertation

AdS/CFT Correspondence with Eight Supercharges

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Abstract

In this thesis we study the AdS_5/CFT_4 correspondence in the presence of eight supercharges, which arises from the consideration of the type IIB superstring theory on $AdS_5 \times X_5$ where X_5 is a Sasaki-Einstein manifold. There are three approaches to the study, namely the four-dimensional superconformal field theory on the boundary, the five-dimensional gauged supergravity in the bulk AdS, and the geometry of the five-dimensional Sasaki-Einstein manifolds, which we review in turn.

One of the principal problems is to find the R-symmetry in the superconformal algebra. It can be done in any of the three approaches above, utilizing the maximization of the central charge a, the minimization of the superpotential P, and the minimization of the volume Z, respectively. We describe them in detail and study their interrelationship.

During the study the determination of the five-dimensional Chern-Simons coupling in the $AdS_5 \times X_5$ spacetime becomes necessary. We will uncover a beautiful topological formula for it, and match the result with the expectations from the field theory. It constitutes a new check of the AdS/CFT correspondence.

Contents

Chapter 1

Introduction

In late 1997, Maldacena proposed a conjecture [1] which relates the four-dimensional $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory and the type IIB superstring on the product of the five-dimensional anti de Sitter (AdS) space and the five-dimensional sphere S^5. Since the four-dimensional theory is believed to be a conformal field theory (CFT), the relation is often called the AdS/CFT correspondence, which has been proved to be an extremely fruitful idea.

It was surprising because it relates the purely field theoretical phenomena in four dimensions to the gravitational ones in five dimensions. It is a partial embodiment of the old idea of realizing the confining dynamics of four-dimensional gauge theory as the theory of strings. Furthermore, it relates the limit of large 't Hooft coupling to the weakly-curved limit of the AdS spacetime, which is a strong-weak coupling duality. It makes the direct check quite hard because the perturbative region of both sides do not agree, whereas if accepted, it makes the study of the non-perturbative region possible through the perturbation of the dual description.

Soon, the conjecture was formulated in a precise way in the works of Gubser, Klebanov and Polyakov [2] and of Witten [3], and then followed a flurry of exciting works exploring various aspects of the correspondence, e.g. the matching of the structure of the gauge invariant operators to the Kaluza-Klein spectrum on S^5 [4], the extension to orbifold field theories or to other spacetime dimensions, the analysis of slightly non-supersymmetric situation, just to name a few. The results obtained in a first year and a half can be found in the review [5].

Of these early works, a most important one for us is the study of the D3-branes probing the conifold by Klebanov and Witten [6]. The near horizon limit is $\mathrm{AdS}_5 \times T^{1,1}$ where $T^{1,1}$ is an S^1 bundle over $S^2 \times S^2$. There, the gauge theory on the branes cannot be found by direct quantization of open strings because of the non-orbifold nature of the conifold. A related point is that the gauge theory becomes superconformal only in the far infrared, and is strongly coupled there. Still, the presence of eight supercharges allowed the determination of the scaling dimensions, and the dual gravity description gave a lot of information of the strongly-coupled gauge theory. One of the most notable extensions was the study of the non-conformal deformation by Klebanov and Strassler [7], which introduced the holographic dual to the confining gauge theory.

Let us return to the conformal case. Then, the theory is a four-dimensional $\mathcal{N} = 1$ superconformal field theory (SCFT) and its gravity dual is governed by a five-dimensional $\mathcal{N} = 2$ gauged supergravity; both have eight supercharges. $\mathcal{N} = 1$ SCFTs are in general strongly coupled. As such, the study of their dynamical properties is a difficult process. However, the presence of a high degree of supersymmetry and the conformal symmetry should help in the analysis. Indeed, the a-maximization, which was discovered in 2003 by Intriligator and Wecht [8], allowed us to identify the R-symmetry in the superconformal algebra from the possible linear combination of global symmetries. The determination of the R-symmetry is essential because it governs the structure of the SCFT. For example, one can easily determine the central charge of the SCFT and the scaling dimensions of chiral primary operators from the knowledge of the R-symmetry. Before the discovery of a-maximization, the number of known four-dimensional strongly-coupled $\mathcal{N} = 1$ SCFTs was quite limited, because in effect we could only treat SCFTs where the R-symmetry can be identified by its conservation alone. The a-maximization allowed us to study many previously unexplored SCFTs, and their interesting dynamics were revealed, see e.g. [9].

It is natural, then, to wonder how the properties of $d = 4$ $\mathcal{N} = 1$ SCFTs are translated to the language of $\mathcal{N} = 2$ supergravities in five dimensions. Their relation was studied by Ferrara and collaborators in [10, 11] soon after Maldacena made the conjecture for the case with maximal supersymmetry. Since it was before the discovery of the a-maximization, they could not proceed very far, but one can find the following interesting passage in the conclusion of the paper [10] :

> The presence of a scalar potential for supergravities in AdS5 allows to study critical points for different possible vacua in the bulk theory ... It is natural to conjecture that these critical points should have a dual interpretation in the boundary superconformal field theory side.

One of the aims of this thesis is to answer the question. As we will see, the dual interpretation is precisely the a-maximization as was shown by the author of the thesis in [12] [1].

A remarkable development happened in the study of the $\mathrm{AdS}_5 \times X^5$ compactification, coincidentally with the advances in the SCFTs by means of the a-maximization. X^5 needs to be a Sasaki-Einstein manifold in order to preserve eight supercharges. For a long time, S^5 and $T^{1,1}$ are the only known smooth Sasaki-Einstein metrics in five dimensions. The situation totally changed in 2003, when a countably-infinite number of new explicit Sasaki-Einstein metrics called $Y^{p,q}$ were found by Gauntlett, Martelli, Sparks and Waldram [13]. The method to obtain corresponding quiver theories was also being devised around the same time, and the construction of the $Y^{p,q}$ quiver gauge theory followed in the work by Benvenuti, Franco, Hanany, Martelli and Sparks [14]. Applying the a-maximization to the quivers thus obtained, they found a complete agreement of the central charge a and the inverse volume of the $Y^{p,q}$ spaces, which is as it should be from the prescription by Gubser, Klebanov, Polyakov [2] and by Witten [3]. It was a check of the AdS/CFT correspondence for a countably-infinite family.

Furthermore, Martelli, Sparks and Yau [15, 16] also found a method to find the geometric counterpart to the R-symmetry by a kind of minimization for the volume of Sasakian manifolds. It was successfully matched to the a-maximization by Butti and Zaffaroni [17] in a somewhat brute-force manner. To review these interesting developments which happened in the recent few years is another objective of the thesis.

These results have convincingly shown the importance of five-dimensional Sasaki-Einstein manifolds, just as the heterotic compactification showed that of Calabi-Yau manifolds. Then, one of the fundamental problems is to carry out the Kaluza-Klein expansion of the ten-dimensional fields on Sasaki-Einstein manifolds to obtain the five-dimensional Lagrangian on the AdS space. The kinetic term for the gauge fields was studied by Barnes, Gorbatov, Intriligator and Wright in [18]. It was extended to the Chern-Simons interactions by Benvenuti, Pando Zayas and the author in [19], in which we obtained a nice topological formula for the Chern-Simons coefficient. It corresponds to the triangle

[1] It might be said that the authors of [10] could have discovered a-maximization long before [8] if they had pursued the problem they noticed at that time; it exemplifies how the research in general is not straightforward.

anomaly under the AdS/CFT correspondence, and it correctly reproduced the result expected from the field theory dual. The purpose of the thesis is to review these fascinating developments to facilitate the further exploration of the AdS/CFT correspondence.

Organization of the thesis

The main part of the thesis starts with a more detailed introduction to the Maldacena conjecture in Chapter 2. Three major players, namely four-dimensional SCFT, supergravity on the five-dimensional AdS space and the Sasaki-Einstein manifold are briefly introduced in turn. Chapter 3 is a review for the 4d SCFT. The a-maximization is explained and is applied to weakly-coupled SCFTs. Next in Chapter 4, we move to the study of the dictionary established by [2, 3] translating the phenomena in AdS to those in CFT and vice versa. Then in Chapter 5, we apply the dictionary to the a-maximization and we will find that it corresponds to the minimization of the superpotential P. We turn to the geometry of the Sasaki-Einstein manifolds in Chapter 6. Two main topics are the description of the newly-found $Y^{p,q}$ spaces and the Z-minimization to find the Reeb vector. Chapter 7 consists of the description of the quiver gauge theory on the D3-branes probing toric Calabi-Yau cones. We treat the $Y^{p,q}$ cases concretely in detail, and briefly state its extension to the generic toric cones. Most of the description up to this point is for the study of the R-symmetry from various guises, while Chapter 8 is devoted to investigate the global internal symmetry as a whole, including their triangle anomalies. We will explain how one can obtain its counterpart in the AdS space, namely the Chern-Simons coefficient, via Kaluza-Klein reduction. We then successfully compare it to the field theory expectation. We show, then, that a-maximization and Z-minimization always agree. We conclude with the summary and the outlook for further research in Chapter 9. The interrelation of the chapters is summarized in figure 1.1.

The material presented in Chapter 5 is based on the author's paper [12], while most of the results in Chapter 8 is taken from the work [19] which is a collaboration of the author with S. Benvenuti and L. A. Pando Zayas. The other material is taken from various sources. The author hopes that this thesis might serve as an introductory review to the fascinating subject of the AdS/CFT correspondence in the presence of eight supercharges.

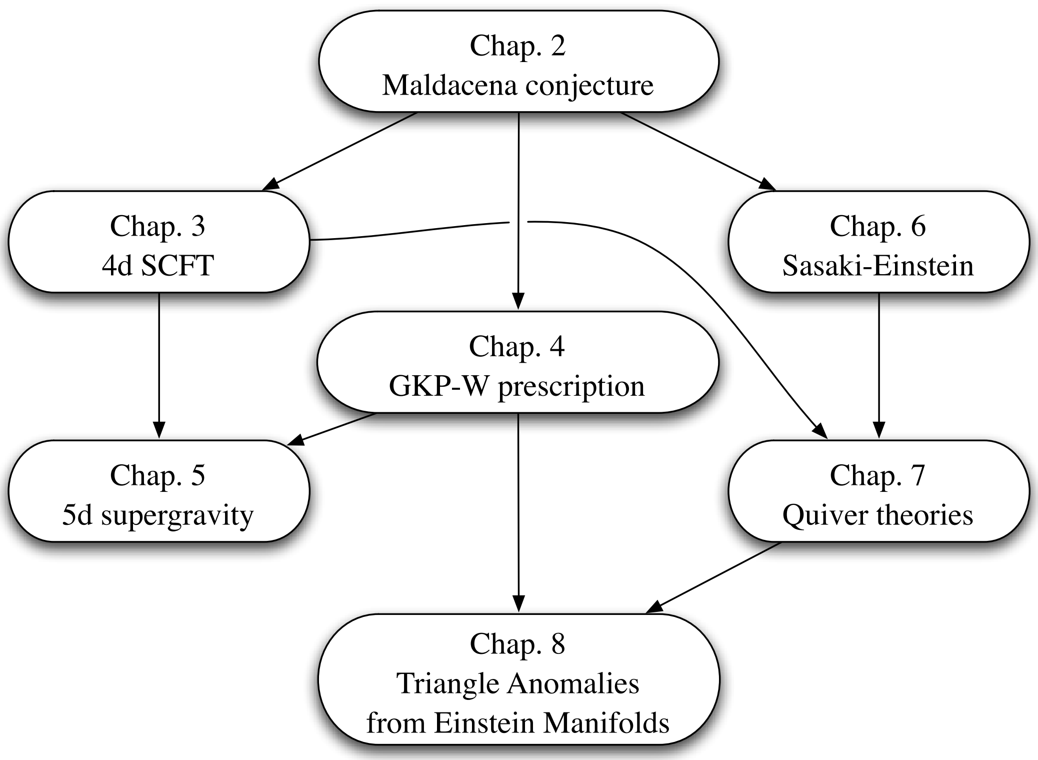

Figure 1.1: Connections of the chapters.

Chapter 2

The Maldacena conjecture

2.1 D3-branes on the flat space

Let us consider the type IIB string theory on the flat ten-dimensional Minkowski space, and introduce a large number, say N, of D3-branes at $x^4 = x^5 = \cdots = x^9 = 0$. In analyzing the system, it is important to bear in mind that D-branes have two descriptions in string theory. One is as the place where open strings can have ends, while the other is as the source of the closed strings. These two are connected by the channel duality of the string theory.

One way to analyze the system is to employ the perturbative viewpoint. Then, the closed string spectrum is the ones for the flat spacetime, and the D3-branes host the excitations of the open strings. They interact through joining and splitting of the string worldsheet. The spectrum includes the whole tower of the massive string states, but when one concentrates on the extreme low energy dynamics of the system, only the zero modes of the open and the closed strings enter.

On the D3-branes, the open string spectrum is then that of the $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theory, where the worldsheet modes X^0 to X^3 provide the gauge fields and X^4 to X^9 yield the six scalar fields. In the ten-dimensional bulk there is the spectrum of the type IIB supergravity. The supergravity sector and the super Yang-Mills sector decouple in the low energy limit, while the coupling among the fields of the super Yang-Mills remains constant in the limit. Thus, we get the following schematic decomposition:

$$\begin{aligned} &\text{D3-branes in flat Minkowski space} \\ &\qquad\qquad \Longrightarrow \text{low energy limit} \Longrightarrow \\ &\qquad\qquad\qquad \text{interacting super Yang-Mills + decoupled 10d supergravity on flat space.} \end{aligned} \tag{2.1.1}$$

Another way of the analysis is to consider the backreaction of the D3-branes to the bulk Minkowski space. As was briefly mentioned, the D3-branes are the source of the closed string fields. Specifically, they source the metric g and the self-dual five-form field $F_5 = dC_4$ in the type IIB supergravity. Before moving forward, we need to set up our convention for the metric and the self-dual five-form field. We normalize F_5 to have $\int F_5 \in 2\pi\mathbb{Z}$. The coupling to the D3-brane is then $S = \int_{D3} C_4$, where the coefficient is fixed so that e^{iS} should have an unambiguous value. D3-branes are their own electromagnetic dual, thus one D3-brane should create five-form flux which satisfies the same quantization condition $\int F_5 \in 2\pi\mathbb{Z}$. Thus, the action for F_5 is fixed to be

$$S_{F_5} = \frac{1}{8\pi} \int_{\mathrm{AdS}\times X} F_5 \wedge *F_5 \tag{2.1.2}$$

where the self-duality is placed by hand after deriving the equation of motion. We take the action for the metric to be

$$\frac{1}{(2\pi)^7\alpha'^4 g_s^2}\int \sqrt{g}R, \tag{2.1.3}$$

as is customary for ten-dimensional string theory.

In the limit where the string coupling g_s is small, the D3-branes have charge and tension proportional to g_s^{-1}, while the coupling constants in the supergravity goes like g_s^2. Thus, the effect of one D3-brane is of order g_s, which becomes negligible in the weak coupling limit. However, when one introduces a huge number N of D3-branes which competes with the small factor g_s so that Ng_s is kept constant, the backreaction remains finite.

The supergravity solution corresponding to such a situation is known, and has the following metric:

$$ds^2 = H(r)^{-1/2}dx_4^2 + H(r)^{1/2}(dr^2 + r^2 d\Omega_5^2), \tag{2.1.4}$$

where $d\Omega_5^2$ is the metric of a five-sphere of radius one, and

$$H(r) = 1 + \frac{4\pi\alpha'^2 g_s N}{r^4} \tag{2.1.5}$$

is a harmonic function on the transverse space $(x^4, x^5, \ldots, x^9)$ which is sourced by the presence of the D-branes. The other nonzero field is the five-form flux which is given by

$$F_5 = (1+*)2\pi N\frac{\mathrm{vol}(S^5)}{\pi^3} \tag{2.1.6}$$

where $\mathrm{vol}(S^5)$ is the volume form of the unit five-sphere S^5. $H(r)$ asymptotes to 1 in the limit $r \to \infty$, which means that the asymptotic infinity of the solution above is the flat ten-dimensional Minkowski space. In the opposite limit of $r \to 0$, $H(r)$ goes as r^{-4}, which in turn implies the presence of a huge redshift factor $r^2 dt^2$ in (2.1.4). Thus, the low energy excitations in this background is either the low energy waves propagating in the flat background, or the modes trapped in the 'near horizon' region $r \sim 0$. In the latter, the energy in their proper coordinate system is of order one but the redshift factor renders it small. Their interaction itself remains finite. Schematically one obtains the decomposition

$$\begin{aligned}&\text{D3-branes as the supergravity background}\\ &\qquad\qquad\Longrightarrow \text{low energy limit} \Longrightarrow\\ &\qquad\qquad\text{interacting modes in the near horizon region}+\\ &\qquad\qquad\qquad\qquad\text{decoupled 10d supergravity on flat space.}\end{aligned} \tag{2.1.7}$$

Let us assume the left hand sides of (2.1.1) and (2.1.7) describes the same physics through open-closed duality. Then one can equate their right hand sides, and one arrives at the conjecture

$$\begin{aligned}&\text{interacting super Yang-Mills on the brane}\\ &\qquad\qquad= \text{interacting modes in the near horizon region.}\end{aligned} \tag{2.1.8}$$

The $r \to 0$ limit of (2.1.4) is

$$\frac{ds^2}{L^2} = \left(u^2 dx_4^2 + \frac{du^2}{u^2}\right) + d\Omega_5^2 \tag{2.1.9}$$

where

$$L^4 = 4\pi\alpha'^2 g_s N \tag{2.1.10}$$

determines the length scale. The terms in the parentheses in (2.1.9) are the metric of the five-dimensional Anti de Sitter (AdS) space, while the other term describes S^5 with constant radius.

Thus we get a conjectured equivalence

$$\mathcal{N} = 4\ SU(N) \text{ super Yang-Mills} = \text{type IIB string on } \mathrm{AdS}_5 \times S^5. \tag{2.1.11}$$

This is the celebrated conjecture of Maldacena [1]. What we provided above in deriving the conjecture is a very crude version of the original one, which was based on many concrete calculations on e.g. the greybody factors of the D3-branes from the viewpoints both of the open string and the closed string. Let us proceed to see how the parameters and the symmetry match on the both sides. Detailed prescription for checking this conjecture will be laid out in chapter 4.

2.2 Translation of parameters

Let us discuss briefly how the parameters on both sides should correspond. The $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory has two parameters, namely the number of colors N and the gauge coupling g, while on $\mathrm{AdS}_5 \times S^5$ side one has the number N of the flux through S^5 and the string coupling g_s. The curvature radius is fixed by (2.1.10). Recalling that the gauge coupling arose from the open string scattering, one has $g_s = g^2$. As for N, the number of colors is the number of D-branes, which turns into the number of flux through S^5, so that two N on both sides should be equated.

Let us investigate more closely. In the gauge theory side it is convenient to employ the double line notation introduced by 't Hooft to do the perturbative calculation. Then the diagrams can be thought to be drawn on a Riemann surface, and the perturbative series can be reorganized so that the string coupling constant which counts the genus of the double-line diagram is $1/N$ and that the worldsheet coupling is given by $\lambda = g^2 N$. On the gravity side, the string coupling constant is none other than g_s and the worldsheet coupling is determined by the spacetime curvature $1/L^4 \propto \lambda^{-1}$. Although the Maldacena conjecture can be thought of as the realization of the idea by 't Hooft of relating four-dimensional Yang-Mills theory to a theory of strings [20], the worldsheet structures on the both sides is totally different, because the worldsheet coupling constants are inversely proportional to each other. It is a strong-weak coupling duality.

This is both a blessing and a misfortune at the same time. It makes the direct check of the conjectural duality quite difficult, because the perturbative regions on both sides do not agree, while if one believes the consistency of the string theory and the argument presented above, it provides a way to analyze strongly-coupled dynamics of the gauge theory using the weakly-coupled gravitational theory, or vice versa.

Before moving to the next section we need to mention the varying opinion on the validity of the conjecture. The argument which led to the conjecture used various intuition on the weakly coupled system. Thus it can be generally assumed that the conjecture holds when both sides reduces to the genus zero part of the dynamics, which requires taking $N \to \infty$ and $g^2 \to 0$ while $\lambda = g^2 N$ is kept fixed. The conjecture in this parameter region is sometimes called the weak form of the conjecture, and that in the finite N region is called the strong form. The literature on the check of the weak form of the conjecture is extremely vast, while that for the strong form is relatively few. In this thesis we will only treat the weak form of the conjecture, that is, we always assume that we are taking the large N limit.

2.3 Matching of symmetries

Before taking the low energy limit, the system has the Poincaré symmetry acting on the coordinates $x^0, \ldots, x^3$. It also has the $SO(6)$ symmetry which rotates $x^4, \ldots, x^9$. From the viewpoint of the $\mathcal{N} = 4$ super Yang-Mills, it is the $SU(4)_R \simeq SO(6)$ R-symmetry. These two symmetries exist before taking the low energy limit.

The near horizon limit makes the spacetime to AdS$_5$. This is a homogeneous space with isometry $SO(4, 2)$ which includes the Poincaré symmetry. Thus the conjecture above predicts the existence of $SO(4, 2)$ symmetry in the low energy limit of the open string theory on the D3-branes. An important fact here is that the four-dimensional conformal symmetry is precisely the group $SO(4, 2)$. Thus it predicts that the low energy limit will be a conformal field theory (CFT). The conjecture by Maldacena is often called the AdS/CFT duality from this reason.

Indeed, when one throws away all the massive modes, the open strings form the $\mathcal{N} = 4$ super Yang-Mills theory, which is classically conformal. One can check easily that the one-loop beta function vanishes, and it is proved that the beta function is zero to all order in perturbation theory. It is also believed that the scale invariance persists even to the non-perturbative level. Thus, on both sides of the open-closed duality the low energy limit leads to the enhancement of the spacetime symmetry from the Poincaré symmetry to $SO(4, 2)$.

One can also compare the supercharges on both sides. Firstly let us analyze the situation before taking the limit. The type IIB superstring has 32 supercharges, while the introduction of parallel D3-branes breaks the supersymmetry in half, resulting in the presence of remaining 16 supercharges. On the open string side, $\mathcal{N} = 4$ supersymmetry means that there are four times the number of supercharges in a spinor of $SO(3, 1)$, which has four real components. Then four times four matches 16 in the counting above. On the closed string side with the brane solution (2.1.4), the number of unbroken supertransformation in the background can be counted, given the supersymmetry transformation law of the gravitino and the dilatino. One can check a half of the supersymmetry remains.

Let us move on to the system after taking the low energy limit. As will be explained in detail in section 3.1.2, the existence of the conformal symmetry doubles the number of supercharges, thus there are 32 supercharges. On the closed string side, one can check that in the AdS$_5 \times S^5$ limit the spacetime curvature term in the supersymmetry transformation law is precisely canceled by the contribution from the self-dual five-form flux. Thus there are 32 supercharges also on the gravity side. The $\mathcal{N} = 4$ supersymmetry algebra is enhanced to the $\mathcal{N} = 4$ superconformal symmetry $SU(2, 2|4)$ on both sides, which includes the internal $SU(4)_R \simeq SO(6)$ symmetry.

2.4 D3-branes on the cone

Vast literature exists in which the conjectured duality between the $\mathcal{N} = 4$ super Yang-Mills and the type IIB superstring on AdS$_5 \times S^5$ is studied in great detail. We would like to move on in another direction, by slightly modifying the starting point of the duality, namely the D3-branes in the flat space. Instead, let us consider a space of the form

$$ds^2 = dx_4^2 + (dr^2 + r^2 d\Omega_5^2) \tag{2.4.1}$$

where dx_4^2 is the metric of the flat four-dimensional Minkowski space and $dr^2 + r^2 d\Omega_5^2$ is the metric of a conic Ricci-flat six-dimensional space Y_6, see figure 2.1. $d\Omega_5^2 = g_{mn} dx^m dx^n$ is the metric for the five-dimensional angular part X_5. The Ricci-flatness of the cone leads to the condition

$$R^{(5)}_{mn} = 4 g^{(5)}_{mn}. \tag{2.4.2}$$

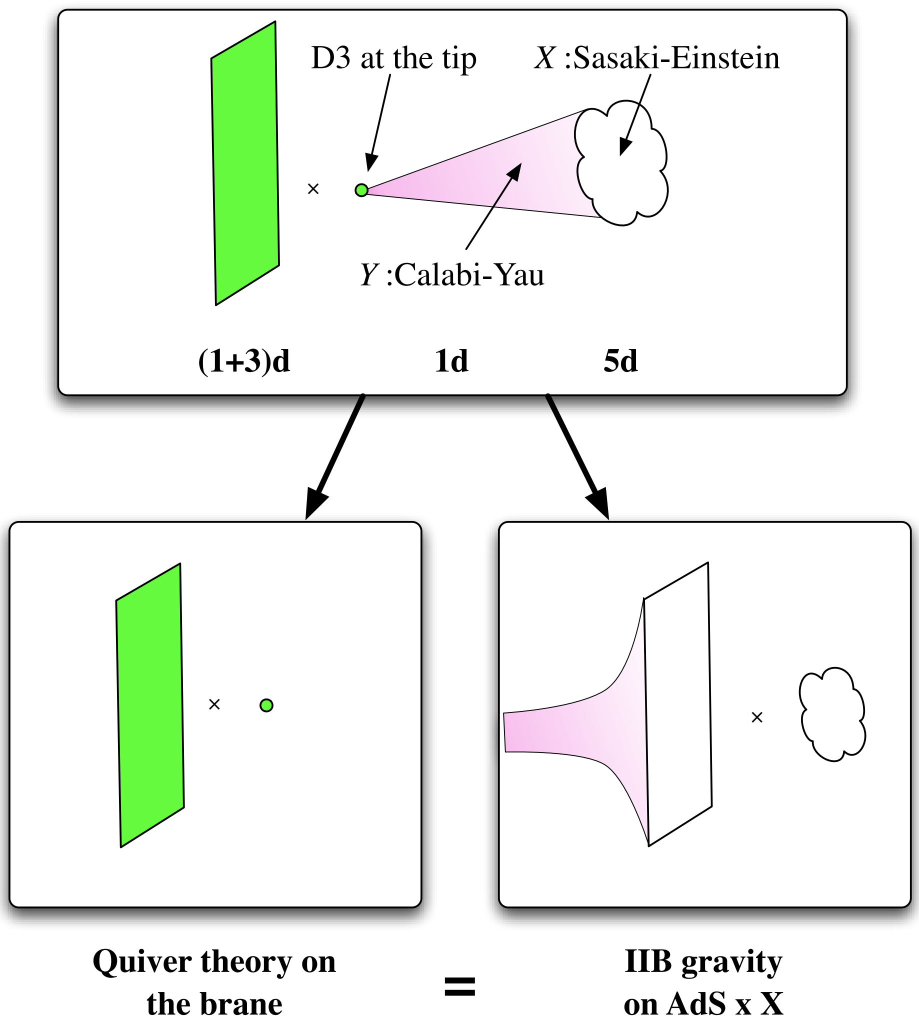

Figure 2.1: Schematic description of the Maldacena conjecture

The proportionality constant 4 is determined by the condition that the angular part X_5 is the slice of the cone at the radius $r = 1$. Ricci flatness ensures that the metric above is the solution of the equation of motion of the type IIB supergravity with all other fields being zero.

Now consider the introduction of a huge number of D3-branes at the tip $r = 0$ of the cone. It is not clear a priori whether it makes sense to put D3-branes to such a singular point, but for the orbifold singularity it is known that one can consistently define the D3-branes at the singularity, and that one can find the perturbative open string spectrum on them. It is also known that one can analyze branes on a non-orbifold singularity. Let us move on, for now, assuming that such a system is well-defined as a string theoretic system.

Then the analysis in section 2.1 goes through almost unmodified. Indeed, (2.1.4) is still the supergravity solution for the stack of D3-branes, with the understanding that $d\Omega_5$ is now the metric of X_5. One can take the near horizon limit as before, and we obtain the space $\mathrm{AdS}_5 \times X_5$, with N units of the self-dual five-form flux through X_5. Thus the following conjectural equivalence emerges:

$$\begin{aligned}&\text{Low energy theory on the D3-branes at the tip of the cone } Y_6\\ &\qquad = \text{the type IIB superstring theory on } \mathrm{AdS}_5\times X_5 \text{ with flux.}\end{aligned} \tag{2.4.3}$$

One point which later becomes important is that the relation (2.1.10) relating g_s, N and the curvature radius L depends on the volume of X in the form

$$L^4 = \frac{4\pi^4\alpha'^2 g_s N}{\mathrm{Vol}(X_5)} \tag{2.4.4}$$

where $\mathrm{Vol}(X_5)$ is the volume of X_5. Let us next look into some of the examples.

2.4.1 Orbifolds

Firstly one can take an orbifold $\mathbb{R}^6/\Gamma$ as the six-dimensional cone Y_6, where Γ is a discrete subgroup of the rotation $SO(6)$. If $\Gamma \subset SU(2) \subset SO(6)$ then there remains $\mathcal{N} = 2$ supersymmetry, and if $\Gamma \subset SU(3) \subset SO(6)$ there is $\mathcal{N} = 1$ supersymmetry in four dimensions. There remain 8 and 4 supercharges, respectively.

On the gravity side the near horizon limit becomes $\mathrm{AdS}_5 \times S^5/\Gamma$. In this limit the number of unbroken supercharges doubles, which results in the remaining 16 or 8 supercharges, depending whether Γ is in $SU(2)$ or $SU(3)$.

On the gauge theory side, one can easily determine the zero modes on the D3-branes on the tip. It uses the method first spelled out in the reference [21], by introducing as many D3-branes as the order $\#\Gamma$ of the orbifolding group Γ on the covering space $\mathbb{R}^6$ and specifying the action of Γ on the Chan-Paton indices as that of the regular representation of Γ. The result is a so-called quiver gauge theory, with $\#\Gamma$ of $SU(N)$ gauge groups and many bifundamental fields which transform in the fundamental representation under one of the gauge groups and in the anti-fundamental representation under another of them. One finds that the perturbative beta function still vanishes in these cases, and one expects that the theory is superconformal. The existence of the conformal symmetry doubles the number of supercharges, thus we get the matching of the number of supercharges between the open- and the closed- string description of the duality.

Let us see an example. Combine six real coordinates $x^4,\ldots,x^9$ in three complex coordinates $z_{1,2,3}$, and take an action of $\mathbb{Z}_2$ generated by

$$(z_1, z_2, z_3) \to (-z_1, -z_2, z_3), \tag{2.4.5}$$

which is in $SU(2) \subset SO(6)$. We expect to get an $\mathcal{N} = 2$ super Yang-Mills theory, and indeed perturbative quantization of the open string leads to the following zero mode spectrum in the $\mathcal{N} = 1$ superfield formalism:

- $W_\alpha^{(1,2)}$: two $SU(N)$ gauge multiplets,
- $\Phi^{(1,2)}$: two chiral superfields in the adjoint representation of the respective $SU(N)$ gauge group,
- $A^{1,2}$: a doublet of chiral superfields transforming in the fundamental under the first $SU(N)$ and in the anti-fundamental under the second $SU(N)$ gauge group, and finally
- $B^{1,2}$: another doublet in the anti-fundamental under the first and in the fundamental under the second $SU(N)$ gauge group.

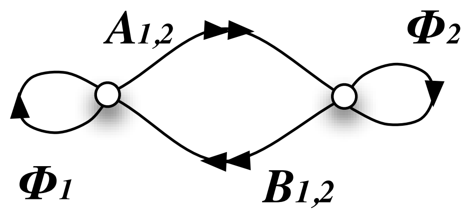

Figure 2.2: Quiver diagram of the $\mathbb{Z}_2$ orbifolded theory.

Fields like $A^{1,2}$ or $B^{1,2}$ are called the bifundamental fields. In the $\mathcal{N} = 2$ formalism $W_\alpha^{(i)}$ and $\Phi^{(i)}$ combine to form an $\mathcal{N} = 2$ vector multiplet, while A^i and B^i combine to form two hypermultiplets. The superpotential is determined from the $\mathcal{N} = 2$ supersymmetry to be

$$W = \sum_{i=1,2} (\operatorname{tr} A^i \Phi^{(1)} B^i + \operatorname{tr} \Phi^{(2)} A^i B^i). \tag{2.4.6}$$

The description becomes the more tedious the more complicated the orbifold action is, hence we introduce a diagrammatic notation as in figure 2.2. There, the nodes denote the $SU(N)$ gauge groups, and the arrow between two nodes represent the bifundamental chiral superfields. The corresponding field transforms as the fundamental under the gauge group in the head of the arrow, and as the anti-fundamental under that in the tail. The arrow which comes back to the same node represents an adjoint chiral superfield of that gauge group. The presence of many arrows in the diagram is the origin of the name 'quiver gauge theory'.

For the $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory, the relation between the six scalar fields $\Phi^{1,\ldots,6}$ and the transverse position of the D3-branes is rather direct. When all the adjoint fields take diagonal vacuum expectation values (vevs), the F-term condition is automatically satisfied, and thus they describe the position of N points in the space $\mathbb{R}^6$ transverse to the D3-branes, spanned by $x^4, \ldots, x^9$. The remaining gauge degree of freedom acts on the diagonal matrices as the Weyl group $\mathfrak{S}_N$ permuting the N diagonal entries, which means these N D3-branes are indistinguishable.

For the $\mathcal{N} = 2$ $SU(N)$ theory things are not as direct. Let us assume that all fields are diagonal, then one can treat each field as a number and the resulting vacua is the N-fold product of the vacuum manifold for the $N = 1$ case divided by $\mathfrak{S}_N$. The F-flatness conditions can be solved by the equations

$$\Phi^{(1)} = -\Phi^{(2)}, \qquad A^1 = B^2, \qquad A^2 = B^1. \tag{2.4.7}$$

At this level the solution is parametrized by $(\Phi^{(1)}, A^1, A^2) \in \mathbb{C}^3$. There is, however, a remaining gauge transformation which leaves the equations (2.4.7) invariant, namely $1 \in U(1)$ for the first node and $-1 \in U(1)$ for the second node. It acts on $\mathbb{C}^3$ by

$$(\Phi^{(1)}, A^1, A^2) \mapsto (\Phi^{(1)}, -A^1, -A^2), \tag{2.4.8}$$

which agrees with the action presented in (2.4.5). Thus we found that the moduli of the theory contains $(\mathbb{C}^3/\mathbb{Z}_2)^N/\mathfrak{S}_N$, which is as it should be as the system of N D3-branes probing the transverse space $\mathbb{C}^3/\mathbb{Z}_2$. The explanation above goes logically backwards in a sense, since the orbifold projection in the quantization of the open strings is implemented so that the resulting moduli space should be the orbifold we started. Nevertheless, we preferred to phrase the exposition in this way since we hoped it illustrates the principle that the supersymmetric gauge theories which is dual to the geometry should have the N-fold product of the original geometry as the moduli space.

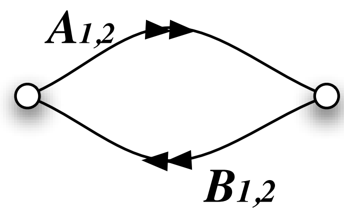

Figure 2.3: Quiver diagram for the conifold theory.

2.4.2 Non-orbifold models

Let us next move on to the non-orbifold cases, which is one of the main topics of this thesis. In order to have a supersymmetric system, we demand the cone Y_6 to be Calabi-Yau, which means that the holonomy group of the manifold reduces to the subgroup $SU(3)$ of the generic $SO(6)$. The six-dimensional spinor representation **4** decomposes as **3** + **1** under $SU(3)$. Then, for a Calabi-Yau manifold one fourth of the original supersymmetry remains. Taking into account the presence of D3-branes, one sees that there exists four supercharges in the system. In passing, we would like to remark that one can get eight remaining supercharges if the holonomy reduces further down to $SU(2)$ to have a hyperkähler cone. It is known however, that such a cone is always an orbifold of the form $\mathbb{R}^2 \times \mathbb{C}^2/\Gamma$ where $\Gamma \subset SU(2)$ is one of the discrete subgroups classified by its ADE type.

A manifold X_5 is called Sasaki-Einstein if the cone over it is a Calabi-Yau cone. Thus, given a Sasaki-Einstein manifold, we get a conjectural duality between the theory on the D3-branes on the tip and the type IIB superstring on $\mathrm{AdS}_5 \times X_5$. We will see the property of the Sasaki-Einstein manifolds in more detail in chapter 6. Here we will touch one of the examples.

Prior to the publication of [13] in 2004, there were only one five-dimensional Sasaki-Einstein manifold with the explicitly known metric except S^5, which is called $T^{1,1}$. It is an S^1 bundle over $S^2 \times S^2$, and has the isometry group $SU(2)_1 \times SU(2)_2 \times U(1)$. The $SU(2)_i$ acts as the rotation of two S^2, while $U(1)$ acts as the shift of the coordinate of the circle fiber. The cone over it is Calabi-Yau, and thus has a natural complex structure. It can be described by the equation

$$x^2 + y^2 + z^2 + w^2 = 0. \tag{2.4.9}$$

The manifold is called the conifold in the string theory literature, and is the local form of one of the simplest singularities that can occur in a compact Calabi-Yau manifold.

The gauge theory on a stack of D3-branes probing the tip of the conifold was constructed in [6] and it has the gauge theory with matter content depicted in figure 2.3, and the superpotential is

$$W = \epsilon_{ij}\epsilon_{ab} \operatorname{tr} A^i B^a A^j B^b. \tag{2.4.10}$$

Let us see how the gauge theory has the properties that match with that of the conifold. $SU(2)_1$ and $SU(2)_2$ act on the index i of A^i and B^i respectively, and the remaining $U(1)$ acts as the R-symmetry where the R-charge of the bifundamentals is uniformly $1/2$, so that the superpotential has the R-charge two. Thus the global symmetries match. Furthermore, the moduli space of the gauge theory reproduces the conifold. Indeed, by taking the bifundamentals to be diagonal, one sees that the F-flatness condition becomes trivial. Let the entries in the diagonal be denoted by $a_{1,2}$ and $b_{1,2}$. One still needs to mod this out by the gauge group, which can be effectively done by forming the gauge invariant observables. They are generated by the combinations

$$s = a_1 b_1, \qquad t = a_1 b_2, \qquad u = a_2 b_1, \qquad v = a_2 b_2, \tag{2.4.11}$$

with one constraint

$$sv = tu. \tag{2.4.12}$$

One can easily see two equations (2.4.9) and (2.4.12) describe the same hypersurface, which was to be shown.

The quiver diagrams fig. 2.2 and fig. 2.3 have many similarities. They share the same global symmetry $SU(2) \times SU(2) \times U(1)$, and indeed the latter can be obtained by adding a mass term

$$m(\mathrm{tr}\,\Phi^{(1)2} - \mathrm{tr}\,\Phi^{(2)2}) \tag{2.4.13}$$

to the Lagrangian of the former and integrating out the adjoint superfields $\Phi^{(1,2)}$. The process has a counterpart in the geometry. Namely, $S^5/\mathbb{Z}_2$ and $T^{1,1}$, when divided by the $U(1)$ symmetry yield $\mathbb{CP}^2/\mathbb{Z}_2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$, respectively. The $\mathbb{Z}_2$ orbifold singularity can be blown up to yield the Hirzebruch surface $\mathbb{F}_2$, which is topologically $S^2 \times S^2$. Thus the transition from $S^5/\mathbb{Z}_2$ to $T^{1,1}$ is done by the blowing up, which is done in the language of the string theory by giving a vev to the twisted sector fields of the orbifold, which can be identified with the operator (2.4.13).

There is a vast literature studying this pair of the gauge theory and the gravity background, including the deformations thereof. The papers [13] and [14] changed the landscape of the field altogether. The first paper constructed a countably-infinite number of new explicit Sasaki-Einstein metrics in five dimension known as the $Y^{p,q}$ spaces, and the second paper constructed corresponding quiver gauge theories. Later, students of the field conducted many consistency checks of the duality, of which the result contained in this thesis is one. Before proceeding in the direction mentioned, we first need to make a preparation of fundamentals in more detail.

Chapter 3

Properties of SCFT_4

The aim of this chapter is to collect important facts about the superconformal algebra in four spacetime dimensions. A lucid exposition on what was known before the discovery of the a-maximization can be found in the report [22]. As for the a-maximization, the definitive reference is still the original article [8].

3.1 Algebras

3.1.1 Conformal algebra

A diffeomorphism f of a manifold is called a conformal transformation if it changes its metric only by an overall factor, i.e.

$$g_{\mu\nu}(x) \to f^* g_{\mu\nu}(x) = e^{\omega(x)} g_{\mu\nu}(x) \tag{3.1.1}$$

for some function $\omega(x)$. Thus it includes any isometry of the manifold. For the Minkowski space $M^{d-1,1}$ it includes the inversion,

$$\mathrm{inv}: \quad x^\mu \mapsto x^\mu/(x^\mu x_\mu) \tag{3.1.2}$$

It is known that in dimension $d > 2$, the Poincaré symmetry and the inversion generates the whole conformal symmetry.

The conformal algebra includes the translation P_μ and the Lorentz rotation $M_{\mu\nu}$ from the isometry. It also includes

$$K_\mu = \mathrm{inv} \cdot P_\mu \cdot \mathrm{inv}, \tag{3.1.3}$$

called the special conformal transformation. Additionally, there is a scalar operator D which is the generator of the dilation

$$x^\mu \mapsto \alpha x^\mu. \tag{3.1.4}$$

Nonzero commutators, in addition to the ones in the Poincaré symmetry, are

$$[D, P_\mu] = -iP_\mu, \tag{3.1.5}$$
$$[D, K_\mu] = +iK_\mu, \tag{3.1.6}$$
$$[M_{\mu\nu}, K_\rho] = -i(\eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu), \tag{3.1.7}$$
$$[P_\mu, K_\nu] = -i(2\eta_{\mu\nu}D + 2M_{\mu\nu}). \tag{3.1.8}$$

D, P_μ, K_μ and $M_{\mu\nu}$ are all Hermitian.

Let us introduce labels $M, N, \dots$ which runs in the range $+$, $-$, and 0 to $d-1$, and rename operators as follows:

$$\Omega_{\mu\nu} = M_{\mu\nu}, \quad \Omega_{+-} = D, \quad \Omega_{\mu\pm} = \frac{1}{2}(P_\mu \pm K_\mu). \tag{3.1.9}$$

The operators Ω_{MN} satisfy the commutation relation of $SO(d,2)$, with the indices $+$ and 0 correspond to the timelike directions and and the rest spacelike. In the following we only treat the case $d = 4$. The algebra $SO(4,2)$ is isomorphic to $SU(2,2)$.

3.1.2 Superconformal algebra

The $\mathcal{N}$-extended superconformal algebra is the algebra generated by the conformal algebra and the $\mathcal{N}$-extended supersymmetry. The latter incorporates the fermionic generators Q^i_α for $i = 1, \dots, \mathcal{N}$ and the anticommutators are

$$\{Q^i_\alpha, Q^{\dagger\bar{j}}_{\dot\beta}\} = 2\delta^{i\bar{j}}\sigma^\mu_{\alpha\dot\beta}P_\mu. \tag{3.1.10}$$

Combination with the conformal algebra leads to the presence of further supercharges $S^{\,i}_{\dot\beta}$ which come from the commutation of K_μ and Q^i_α. The $U(\mathcal{N})$ R-symmetry $T^{i\bar{j}}$ is also generated where the indices i and $\bar{j}$ transform under the $U(\mathcal{N})$ symmetry in the fundamental and the anti-fundamental representations, respectively. We denote the $U(1)_R \subset U(\mathcal{N})_R$ part by $R = T^{i\bar{i}}/\mathcal{N}$.

The nonzero commutation relations, which are not fixed by the super-Poincaré and $SU(\mathcal{N})_R$ symmetry alone, are

$$[D, Q^i_\alpha] = -\frac{i}{2}Q^i_\alpha, \qquad [D, S^{\,i}_{\dot\beta}] = +\frac{i}{2}S^{\,i}_{\dot\beta}, \tag{3.1.11}$$

$$[R, Q^i_\alpha] = -Q^i_\alpha, \qquad [R, S^{\,i}_{\dot\beta}] = +S^{\,i}_{\dot\beta}, \tag{3.1.12}$$

$$[K_{\alpha\dot\beta}, Q^i_\gamma] = 2i\epsilon_{\alpha\gamma}S^{\,i}_{\dot\beta}, \qquad [P_{\alpha\dot\beta}, S^{\,i}_{\dot\gamma}] = 2i\epsilon_{\dot\beta\dot\gamma}Q^i_\alpha, \tag{3.1.13}$$

while the extra anti-commutators are

$$\{S^{\,i}_{\dot\beta}, S^{\dagger\bar{j}}_{\ \alpha}\} = 2\delta^{i\bar{j}}\sigma^\mu_{\dot\beta\alpha}K_\mu, \tag{3.1.14}$$

$$\{Q^i_\alpha, S^{\dagger\bar{j}}_{\ \beta}\} = \delta^{i\bar{j}}\epsilon_{\alpha\beta}(2iD - \mathcal{N}R) + \delta^{i\bar{j}}M_{\alpha\beta} + 4\epsilon_{\alpha\beta}T^{i\bar{j}}. \tag{3.1.15}$$

The relation (3.1.15) is fixed by the Jacobi identity, say,

$$[Q, \{S^\dagger, Q^\dagger\}] + [S^\dagger, \{Q^\dagger, Q\}] + [Q^\dagger, \{Q, S^\dagger\}] = 0. \tag{3.1.16}$$

The algebra is isomorphic to the superalgebra $SU(2,2|\mathcal{N})$, with its bosonic part $SU(2,2) \times SU(\mathcal{N})_R \times U(1)_R$. The $U(1)_R$ generator acts on $\mathbb{C}^{2,2|\mathcal{N}}$ as the matrix

$$\mathrm{diag}(\underbrace{\mathcal{N}, \dots, \mathcal{N}}_{4\text{ times}} \mid \underbrace{4, \dots, 4}_{\mathcal{N}\text{ times}}) \in SU(2,2|\mathcal{N}) \tag{3.1.17}$$

which is proportional to the identity if $\mathcal{N} = 4$. Thus the $U(1)_R$ decouples from the superconformal algebra for $\mathcal{N} = 4$. It can also be seen by contracting (3.1.15) with $\delta_{i\bar{j}}$. Indeed, the $U(1)$ R-symmetry appears with coefficient $\mathcal{N}(4 - \mathcal{N})$ in the contraction, which vanishes for $\mathcal{N} = 4$. When the $\mathcal{N} = 4$ algebra is decomposed with respect to the $\mathcal{N} = 1$ subalgebra, the $U(1)_R$ symmetry of the subalgebra will be an element of the Cartan subalgebra of $SU(4)_R$. In the rest of the thesis, we only treat $\mathcal{N} = 1$ superconformal algebra. Then, (3.1.15) simplifies to

$$\{Q_\alpha, S^\dagger{}_\beta\} = \epsilon_{\alpha\beta}(2iD + 3R) + M_{\alpha\beta}. \tag{3.1.18}$$

3.2 Multiplets

3.2.1 Unitary representations of the conformal symmetry

The Hilbert space of a conformal field theory is acted by the conformal group. The gauge invariant part of the Hilbert space, in particular, should have a decomposition into the direct sum of positive energy irreducible unitary representations (irreps) of the conformal group. Thus it is of fundamental importance to find which kind of irreps is possible, just as in the ordinary field theory. There, the classification of the irreps of the Poincaré group is the first step in the analysis of the theory. The determination of the unitary irreps of the conformal algebra was carried out in detail in [24]. We present only the summary in the following.

Consider an irrep of the conformal algebra, and decompose it under the subgroup $SO(4)\times SO(2) \subset SO(4,2)$, where $SO(2)$ is generated by the dilatation D. The off-diagonal components of $SO(4,2)$ can be arranged to constitute the raising and lowering operators. Denote them by P'_m and K'_m respectively, where m is the index of the vector under $SO(4)$. P'_m and K'_m are Hermitian conjugates of each other. The positivity of energy can be used to show that the dilatation eigenvalue, or the dimension d, is bounded below. Application of the lowering operators eventually leads to the lowest dimension operators in the irrep, which is annihilated by them. They are called the conformal primary states.

Then any states in the irrep with higher dimension, called the descendant states, can be constructed from the primary states by the repeated action of the raising operators. The action of the conformal group on the primaries and descendants is determined by the $SO(4) \times SO(2)$ representation of the primaries. Thus any unitary positive energy irrep is labeled by the dimension d and the two spins s_1, s_2 for $SO(4)$, and the label specifies the irrep uniquely if existed. Let us denote such a representation by $D(d; s_1, s_2)$.

Not all of such label is allowed as a unitary representation, because for a certain combination of d, and $s_{1,2}$, the commutation relation necessarily leads to the existence of the negative norm states. For example, for an irrep $D(d; 0, 0)$ with lowest weight vector $|v\rangle$ of norm one, the vector $P'_m P'_m|v\rangle$ in the level two state necessarily has the norm $8d(d-1)$. Thus, $d = 0$ or $d \geq 1$ is required for the positivity of the Hilbert space. Moreover, $P'_m P'_m|v\rangle = 0$ if $d = 1$, because its norm is zero. Since $P'_m P'_m$ is roughly the d'Alembertian, it implies that $|v\rangle$ is a free field. Similar analysis can be carried out for other irreps, and the end result is that the representation $D(d; s_1, s_2)$ exists as a unitary representation if and only if

1. $d = s_{1,2} = 0$, which is the vacuum representation, or
2. $s_1 s_2 = 0$, $d \geq s_1 + s_2 + 1$, or
3. $s_1 s_2 \neq 0$, $d \geq s_1 + s_2 + 2$.

3.2.2 Unitary representations of superconformal symmetry

For a superconformal theory, the decomposition of the Hilbert space into the irreps of the superconformal group consists the first step of the analysis, which was carried out in the papers [25, 26]. It can be done by employing the decomposition further into the irreps of the conformal group. The references [23, 27] contain a clear exposition of the results. We present the $\mathcal{N} = 1$ case only.

The $\mathcal{N} = 1$ superconformal group $SU(2,2\,|\,1)$ contains $SO(4) \times SO(2)_D \times U(1)_R$ as a maximal subgroup. Extra elements of the superconformal group can be arranged in the raising and lowering operators as before. The states with lowest dimension in an irrep is called the superconformal primary states, and the representation they form under the subgroup $SO(4) \times SO(2)_D \times U(1)_R$ determines the irrep uniquely. Let us label them by $\mathcal{D}(d; s_1, s_2; r)$. They can be decomposed into the sum of irreps of $SU(2, 2)$ which are connected by the action of S and Q operators. Denote the raising and lowering fermionic operators schematically by Q'_a, $Q'_{\dot a}$, S'_a and $S'_{\dot a}$, respectively. They are linear combinations of the unprimed counterparts.

The representation $\mathcal{D}(d; s_1, s_2; r)$ is unitary if and only if

1. $d = r = s_{1,2} = 0$, which is the vacuum, or
2. $d \geq 2s_1 - \frac{3}{2}r + 2$ and $d = +\frac{3}{2}r$ for $s_1 \geq 0$ and $s_2 = 0$, or
3. $d = -\frac{3}{2}r$ and $d \geq 2s_2 + \frac{3}{2}r + 2$ for $s_1 = 0$ and $s_2 \geq 0$, or
4. $d \geq 2s_1 - \frac{3}{2}r + 2$ and $d \geq 2s_2 + \frac{3}{2}r + 2$.

When the inequality is saturated in the case 2 and 3, the field becomes free. In the case 4, the multiplet becomes shorter when any of the equality is attained.

One representation which will be our focus later is the case 2 with $s_1 = s_2 = 0$. Suppose a superconformal primary state $|v\rangle$ is spinless. Then, the positivity of the norm of $Q'_a|v\rangle$ implies

$$D \geq \frac{3}{2}R, \tag{3.2.1}$$

which follows when one applies the commutation relation (3.1.15). The inequality is saturated if $Q'_a|v\rangle$ vanish. We assumed that $|v\rangle$ is a superconformal primary, thus $S'_a|v\rangle = S'_{\dot a}|v\rangle = 0$. Since Q_α is a linear combination of S'_a and Q'_a, we obtain $Q_\alpha|v\rangle = 0$. The state annihilated by Q_α is called chiral, so $|v\rangle$ is called the chiral primaries. For them, the dimension D and the R-charge R is related by the equation $D = 3R/2$.

Another important representation is $\mathcal{D}(2; 0, 0; 0)$ which contains a dimension-two scalar and a conserved current. It satisfies the shortening condition in the case 4 above, and the shortening is the direct consequence of the conservation of the current. The dimension of the scalar field is thus protected to be exactly equal to two.

3.3 Central charges

A particularly important quantity characterizing a conformal field theory in any spacetime dimension is the central charge. For two-dimensional conformal field theories, the central charge c is defined to be the leading coefficient for the operator product expansion (OPE) of the energy momentum tensor with itself

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) + \cdots . \tag{3.3.1}$$

c is positive for unitary theories, adds up if we combine two decoupled CFTs, and is 1 for the free CFT with one bosonic scalar field. Thus c can be said to 'count' the number of freedom in CFT. It can also be measured by the trace anomaly caused by the coupling to the external gravitational field, i.e.

$$\langle T^\mu_\mu \rangle = -\frac{c}{12}R \tag{3.3.2}$$

where R is the scalar curvature of the metric. Furthermore, in two dimensions one can relate the central charge c defined above to the asymptotic density of states using the modular invariance of the torus partition function with the result

$$\log \#(\text{states with dimension } D\,) \;\propto\; \sqrt{cD}. \tag{3.3.3}$$

Thus in two dimensions there is a direct connection with the central charge c to the asymptotic density of states.

The analysis in higher dimensional conformal theory of the central charge as defined by the trace anomaly in an external metric was done e.g. in reference [28], where it was shown that $\langle T^\mu_\mu \rangle$ is a linear combination of the Euler density and of the scalar quantities constructed from the Weyl tensor.

Here the Euler density E in $2d$ spacetime dimensions is defined by

$$E_{(2d)} \propto \epsilon^{i_1 j_1 \cdots i_d j_d} \epsilon^{k_1 l_1 \cdots k_d l_d} R_{i_1 j_1 k_1 l_1} \cdots R_{i_d j_d k_d l_d}, \tag{3.3.4}$$

and the Weyl tensor W_{ijkl} is defined by subtracting trace parts from the Riemann curvature tensor so that any contraction of two indices gives zero. Earlier results concerning the Weyl anomaly in various dimension was lucidly summarized in [29][1].

Thus in four dimensional conformal field theory there are two central charges c and a defined as the coefficients appearing in the equation

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2}(W_{\mu\nu\rho\sigma})^2 - \frac{a}{16\pi^2}E_{(4)} \tag{3.3.5}$$

where

$$E_{(4)} = \frac{1}{4}\epsilon^{\mu_1\nu_1\mu_2\nu_2}\epsilon^{\rho_1\rho_1\rho_2\sigma_2} R_{\mu_1\nu_1\rho_1\sigma_1} R_{\mu_2\nu_2\rho_2\sigma_2}. \tag{3.3.6}$$

Let us next discuss the case for the $\mathcal{N} = 1$ superconformal theory in four spacetime dimensions. Now the energy-momentum tensor is combined with the R-symmetry current and the supertranslation current to form the supercurrent $R_{\alpha\dot\alpha}$, whose lowest component is the R-current itself. In references [30, 31, 32, 33] it was shown that the anomaly for $R_{\alpha\dot\alpha}$ by the external fields can be summarized by the equation

$$\bar{D}^{\dot\alpha} R_{\alpha\dot\alpha} = \frac{1}{24\pi^2}(c\mathcal{W}^2 - a\Xi) \tag{3.3.7}$$

where $\mathcal{W}$, Ξ are the superfields which contain the Weyl tensor and the Euler density in the appropriate places. The same problem was studied from the component formalism in [34] in which the three-point correlator of the supercurrent

$$\langle R_\mu(x_1, \theta_1, \bar\theta_1) R_\nu(x_2, \theta_2, \bar\theta_2) R_\rho(x_3, \theta_3, \bar\theta_3) \rangle \tag{3.3.8}$$

was found to be expressible in a linear combination of two superconformal invariants.

The discussion above implies that the central charges a and c for a superconformal theory is a linear combination of $U(1)_R^3$ and $U(1)_R$-gravity-gravity anomalies. The coefficients can be fixed by considering the CFT consisting of free chiral and vector multiplets, with the result

$$a = \frac{3}{32}(3\,\mathrm{tr}\,R^3 - \mathrm{tr}\,R), \qquad c = \frac{1}{32}(9\,\mathrm{tr}\,R^3 - 5\,\mathrm{tr}\,R). \tag{3.3.9}$$

Here $\mathrm{tr}\,R^3$ and $\mathrm{tr}\,R$ denote the coefficient of the respective three point functions of currents, normalized so that they equal for renormalizable theories to the trace of charge matrix over the label for the Weyl fermions.

[1] It seems to us that their result assumes that the conformal theory at least has CP symmetry in order to eliminate the parity odd operators from appearing in the anomaly. Many of the theories we treat in this thesis are not invariant under parity, but they all have the invariance under CP. Thus, the analysis presented here is applicable.

3.4 a-maximization

Let us consider an $\mathcal{N} = 1$ SCFT in four dimensions with global internal symmetries in addition to the superconformal symmetry. Let us denote by J_I^μ, $(I = 1, 2, \ldots, n_V)$, the currents of non-anomalous global symmetries of the theory and by Q_I corresponding charges. We demand the charges Q_I to be integral, so that J_I can be coupled to external $U(1)^{n_V}$ connections as follows:

$$S \to S + J_I^\mu A_\mu^I + \cdots . \tag{3.4.1}$$

Some of them may rotate the supercoordinates θ_α. Let the charges of θ_α under the global symmetry Q_I be given by $\hat{P}_I$:

$$\theta_\alpha \longrightarrow e^{i\phi^I Q_I}\theta_\alpha = e^{i\phi^I \hat{P}_I}\theta_\alpha \tag{3.4.2}$$

It corresponds to the commutation relation

$$[Q_I, Q_\alpha] = -\hat{P}_I Q_\alpha. \tag{3.4.3}$$

A global symmetry $t^I Q_I$ which commutes with θ_α is called a flavor symmetry. The condition is given by

$$t^I \hat{P}_I = 0. \tag{3.4.4}$$

Global symmetries, even if they are non-anomalous, may have chiral anomalies among them. This can be expressed by saying that the gauge transformation of the external $U(1)^N$ gauge field will have gauge anomaly given by the descent construction starting from the anomaly polynomial

$$\frac{1}{24\pi^2}\hat{c}_{IJK}F^I F^J F^K \tag{3.4.5}$$

where $F^I = F^I_{\mu\nu}dx^\mu \wedge dx^\nu/2$ is the curvature two-form of the I-th external $U(1)$ gauge field. The constants $\hat{c}_{IJK}$ are given by

$$\hat{c}_{IJK} = \operatorname{tr} Q_I Q_J Q_K \tag{3.4.6}$$

where the trace is over the labels of the Weyl fermions of the theory as before. There may also be gravitational anomaly given by

$$\hat{c}_I F^I \operatorname{tr} RR \tag{3.4.7}$$

where R is the curvature two-form of the external metric.

The $U(1)_R$ symmetry R_{SC} which appears in the anticommutator (3.1.18) of the supertranslation Q_α and the special superconformal transformation S^α is a particular combination of global symmetries Q_I so that

$$R_{SC} = r^I Q_I. \tag{3.4.8}$$

We normalize r^I so that the charge of θ_α under $r^I Q_I$ be one, that is, $r^I \hat{P}_I = 1$. We call R_{SC} the superconformal R-symmetry if emphasis is necessary. Any symmetry which rotates the supercharge is called an R-symmetry in the literature on supersymmetric theories, thus some care in the distinction is required.

R_{SC} can be used to uncover many physical properties of the theory considered, as was discussed in the previous subsections. One is that the dimension and the R-charge of the scalar chiral primary is proportional, see (3.2.1). Another relation is with the central charges of the theory, see (3.3.9).

Suppose that some high-energy description of the (possible) SCFT is given. One can identify the non-anomalous symmetry and can calculate $\hat{c}_{IJK}$ by the application of 't Hooft's anomaly matching. The charges $\hat{P}_I$ of θ_α under Q_I will also be easily determined. Then, the basic problem is the identification of the superconformal R-symmetry $R_{SC} = r^I Q_I$.

Here comes the brilliant idea first introduced by Intriligator and Wecht[8]. Let $Q_F = t^I Q_I$ be a flavor symmetry, i.e. $t^I \hat{P}_I = 0$, or $[Q_F, Q_\alpha] = 0$. They showed that the triangle diagram with one Q_F and two R_{SC} insertions can be mapped, by using the superconformal transformation, to the triangle diagram with Q_F and two energy-momentum tensor insertions. One way to express this is by the Konishi current K_F [35] which is the supersymmetric completion of the flavor current J_F. Its anomaly in the presence of external supergravity sources is known to be

$$\bar{D}^2 K_F = \frac{k_F}{384\pi^2} \mathcal{W}^2 \tag{3.4.9}$$

where $\mathcal{W}$ is the same multiplet which appeared in (3.3.7). When expanded, it implies that the magnitude of the Q_F-R-R anomaly and of the Q_F-$T^{\mu\nu}$-$T^{\mu\nu}$ anomaly is determined by a unique constant k_F. Another way to give the same argument is that the three point function

$$\langle K_F(x_1, \theta_1, \bar{\theta}_1) R_\mu(x_2, \theta_2, \bar{\theta}_2) R_\nu(x_3, \theta_3, \bar{\theta}_3) \rangle \tag{3.4.10}$$

in a superconformal field theory is determined by the superconformal algebra up to one overall constant [34].

The precise coefficient can be fixed using a free SCFT. The result is

$$9 \operatorname{tr} Q_F R_{SC} R_{SC} = \operatorname{tr} Q_F. \tag{3.4.11}$$

Another requirement is the negative definiteness

$$\operatorname{tr} Q_F Q_F R_{SC} < 0 \tag{3.4.12}$$

which comes from the positivity of the two point function of currents $\langle J_F(x_1) J_F(x_2) \rangle$.

Let us introduce the trial a-function $a(s)$ for a trial R-charge $R(s) = s^\Lambda Q_\Lambda$ which is defined by the formula

$$a(s) = \frac{3}{32}(3 \operatorname{tr} R(s)^3 - \operatorname{tr} R(s)), \tag{3.4.13}$$

imitating (3.3.9). The conditions (3.4.11), (3.4.12) mean that $a(s)$ is locally maximized at the point $s^I = r^I$, where the trial R-charge becomes the superconformal R-charge R_{SC}. Thus the procedure is called the a-maximization in the literature. It is understood that s is constrained so that the charge of θ_α under $R(s)$ is one.

One easy consequence is that non-abelian non-R symmetry does not mix in the superconformal R-symmetry. Indeed, consider a generator $X \in H \subset G$ in the Cartan subalgebra H of G. In a-maximization we introduce a parameter s and consider the linear combination $R_0 + sX$. The crucial point is that there exists an element $g \in G$ which flips the sign of X, that is, which sends $X \mapsto -X$ and it is a discrete symmetry of the theory. Thus the trial a-function is even in s. It is at most cubic, thus it is an even quadratic function in s which has its only extremum at $s = 0$.

3.5 Weak-coupling examples

We will mainly apply the a-maximization to strongly-coupled theories which have no other means of study. Before embarking on that subject, it would be satisfying to check explicitly that the a-maximization procedure correctly reproduces known results if the conformal point is at the weak coupling in the large N limit and the perturbation theory can be applied. Such a conformal theory was first studied in [36], and the superconformal version was first studied from other perspectives in [37].

The theory we treat is the $\mathcal{N} = 1$ super Yang-Mills theory with $SU(N)$ as the gauge group, with N_F fundamental flavors $(Q, \tilde{Q})$, and one chiral superfield Φ in the adjoint representation . The superpotential is taken to be zero. The theory is asymptotically free if

$$x \equiv \frac{N_c}{N_F} > \frac{1}{2}. \tag{3.5.1}$$

The beta function becomes very small when x approaches the lower bound, and if $x = 1/2$ the theory becomes conformal for any sufficiently small gauge coupling constant.

If the low energy limit of the theory is an SCFT, the superconformal R-symmetry is conserved. Thus it should rotate various fields as

$$W_\alpha \to e^{i\theta} W_\alpha, \qquad Q \to e^{i(1-(1-s)x)\theta} Q, \qquad \Phi \to e^{i(1-s)\theta}\Phi \tag{3.5.2}$$

where s cannot be determined by the conservation of the current.

The determination of s from the a-maximization is straightforward as long as the result does not violate the unitarity bound discussed in sec. 3.2.2. The case including the unitarity bound hit was done in [9], but we limit ourselves to the perturbative cases. Then there are no complication because any gauge invariant operator should have bare dimension ≥ 2, and the anomalous dimension is small because of the weak coupling. The calculation is straightforward and one obtains the result

$$s = 1 - \frac{10}{3(3 + \sqrt{20x^2 - 1})} = \frac{1}{3} + \frac{2}{3}\epsilon - \frac{5}{6}\epsilon^2 + \cdots \tag{3.5.3}$$

where $\epsilon = x - 1/2$. When $\epsilon = 0$ then the R-charges of the chiral superfields all become $2/3$. This is as it should be, because then the theory is conformal at zero coupling and all the fields are free. The fact forces any chiral superfield to have dimension 1 and R-charge $2/3$, as discussed in the previous sections.

Let us now see that the result above agrees with perturbative calculation to the lowest order. The one-loop anomalous dimension is given by

$$\gamma(Q) = -\frac{g^2}{8\pi^2}\frac{N_c^2}{N_F} + O(g^4), \tag{3.5.4}$$

$$\gamma(\Phi) = -\frac{g^2}{8\pi^2} N_c + O(g^4), \tag{3.5.5}$$

while the Novikov-Shifman-Vainshtain-Zakharov beta function is [38]

$$\beta(g) = -\frac{g^3}{4\pi^2}\frac{3N_c - N_F(1-\gamma(Q)) - N_C(1-\gamma(\Phi))}{1 - g^2 N_c/8\pi^2}. \tag{3.5.6}$$

Then we have

$$\frac{g^2}{8\pi^2} = 4\epsilon + O(\epsilon^2) \tag{3.5.7}$$

at the superconformal point, which means

$$D(\Phi) = 1 + \frac{1}{2}\gamma(\Phi) = 1 - \epsilon + O(\epsilon)^2. \tag{3.5.8}$$

Thus

$$s = \frac{1}{3} + \frac{2}{3}\epsilon + O(\epsilon^2), \tag{3.5.9}$$

which matches the result from the a-maximization.

The check given above up to one-loop between the a-maximization and the perturbative calculation can be easily generalized to arbitrary weak-coupling superconformal field theories. It is known that the explicit calculation at two-loops also always agrees.

3.6 a-maximization in the presence of anomalous currents

We saw in the previous section how the superconformal R-symmetry can be found as the combination of non-anomalous global currents by means of the a-maximization. In [39, 40], Kutasov and collaborators incorporated anomalous global currents to the picture. It starts with the same trial a-function

$$a(s) = \frac{3}{32}(3\,\mathrm{tr}(s^I Q_I)^3 - \mathrm{tr}\, s^I Q_I), \tag{3.6.1}$$

where Q_I now include all the global symmetries, anomalous or non-anomalous. Let us denote the a-th gauge fields by $F^a_{\mu\nu}$ and the Adler-Bell-Jackiw anomaly coefficient of the I-th global symmetry with a-th gauge field by m^a_I so that

$$\partial_\mu J^\mu_I \propto \sum_a m^a_I \epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\, F^a_{\mu\nu} F^a_{\rho\sigma}. \tag{3.6.2}$$

We need to impose the anomaly-free condition for each gauge group in the field theory considered. Thus we have to introduce the Lagrange multipliers λ_a and consider

$$a(s,\lambda) = a(s) + \lambda_a(m^a_I s^I). \tag{3.6.3}$$

We need to extremize it with respect to both s^I and λ_a. Define the function $a(\lambda)$ by first maximizing $a(s,\lambda)$ with respect to s^I, fixing λ_a. When $\lambda = 0$, the anomaly free condition is not imposed, and the a-function takes the value for the free field theory. This corresponds to zero gauge coupling. When λ attains the value $\tilde{\lambda}$ where $a(\lambda = \tilde{\lambda})$ is extremized, a becomes the true central charge of the SCFT. In [39] it was shown that a generically decreases along the flow of λ from zero to $\tilde{\lambda}$, suggesting that λ and the gauge coupling constants can be somehow identified. Indeed, there is a certain renormalization scheme where such identification is exact [40]. In [39, 40] Lagrange multipliers were also introduced for the conditions that the terms in the superpotential should have R-charge two. Then, the Lagrange multipliers can be identified with the running coefficients.

Chapter 4

Dictionary for the correspondence

The objective of this chapter is to establish the dictionary which translates the phenomena on the CFT side to those on the AdS side and vice versa.

4.1 Anti de Sitter space

First, we would like to review in more detail the geometry of the Anti-de Sitter space. Consider a hypersurface

$$-X_-^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_+^2 = 1, \tag{4.1.1}$$

in the flat six-dimensional space with the metric $(--++++)$. The surface obviously has the isometry $SO(4,2)$, which is isomorphic to the four-dimensional conformal group, and is an Einstein manifold with negative curvature which satisfies

$$R_{\mu\nu} = -4g_{\mu\nu}. \tag{4.1.2}$$

If the right hand side of (4.1.1) is replaced by L^2, the relation above changes to $R_{\mu\nu} = -4g_{\mu\nu}/L^2$. We call L the curvature radius of the space. It is the solution of equation of motion of the five-dimensional Einstein-Hilbert action

$$S = \frac{1}{2}\int \sqrt{g}\,(R + 12\Lambda) \tag{4.1.3}$$

with $\Lambda = L^{-2}$.

Change the coordinates to those defined by

$$X_- = \cos\tau\cosh\rho, \quad X_0 = \sin\tau\cosh\rho, \quad X_{1,2,3,+} = \omega_{1,2,3,+}\sinh\rho \tag{4.1.4}$$

where $\sum\omega_i^2 = 1$ is the coordinate of the unit three-sphere. The direction along τ is a closed time-like curve. We remove the identification $\tau \sim \tau + 2\pi$, and the resulting space is the five-dimensional Anti-de Sitter space (AdS_5). The coordinate system above is called the global coordinates, and it shows the structure of AdS_5 as $\mathbb{R}\times B^4$ where $\mathbb{R}$ is the time direction τ and B^4 is the four-dimensional ball with the hyperbolic metric.

Another important coordinate system is the Poincaré coordinates, which are defined by the re-parametrization

$$X_{0,1,2,3} = ux^{0,1,2,3}, \qquad X_\pm = \frac{1 \pm (x^i x_i - 1)}{2u}. \tag{4.1.5}$$

The metric in these coordinate system is

$$\frac{du^2}{u^2} + u^2 ds_4^2 \tag{4.1.6}$$

where ds_4^2 is the metric of the flat four-dimensional Minkowski space. This is the form we quoted in (2.1.9). The dilatation of the four coordinates $x^\mu \to cx^\mu$ can be lifted up to the isometry of the Poincaré coordinates

$$x^\mu \to cx^\mu, \qquad u \to c^{-1}u. \tag{4.1.7}$$

The orbit of this action is timelike and non-compact. Thus one can take the orbit as the time direction in AdS.

In the AdS/CFT correspondence one identifies the isometry of AdS and the conformal symmetry of the four-dimensional theory. The generator of the dilatation is that for $SO(2)$ in the subgroup $SO(4) \times SO(2) \subset SO(4,2)$. Thus, D acts on the coordinates $X_{\pm,0,1,2,3}$ as the rotation of X_- and X_0. In the global coordinates, it is the shift in τ, which is the time direction. Thus, the scaling dimension of the operators of the CFT, which is the eigenvalue of the dilatation, corresponds to the energy in the global coordinate of the AdS space which is the eigenvalue of the time translation there.

4.2 GKP-W prescription

Let us next establish a precise relation between the property of the theory on the AdS space and that of the four-dimensional CFT we are considering, which was proposed by Gubser, Klebanov and Polyakov [2] and by Witten [3]. We will abbreviate it as the GKP-W prescription.

The intuition behind it is that the composite operator O from the open string degrees of freedom couples to a closed string mode ϕ, so that we can expect a one-to-one map between the operator in the CFT and the field in the AdS space with a coupling between the two on the brane. The prescription is, then, summarized by the formula

$$Z_{\rm AdS}[\phi\big|_{u=\infty} = \hat{\phi}] = \langle e^{-\int d^4x O(x)\hat{\phi}(x)}\rangle_{\rm CFT}. \tag{4.2.1}$$

The right hand side is the partition function of the CFT in the presence of the source $\hat{\phi}(x)$ for the operator $O(x)$, that is, the generating function of the correlators of $O(x)$. The left hand side is the partition function of the theory in AdS, with the boundary value of ϕ is fixed to be equal to $\hat{\phi}$ which was used as the source. One interesting point is that the universal existence of the energy-momentum tensor on the CFT side implies the existence of a two-index symmetric tensor field $g_{\mu\nu}$ on the AdS side. The conservation of the energy-momentum translates under the prescription above to the diffeomorphism invariance of $g_{\mu\nu}$. Thus, the bulk theory on the AdS is necessarily with the general relativity.

Another point to note is that both sides will need the renormalization. Indeed, short distance singularities of the correlators necessitates the renormalization of the couplings on the right hand side, while the non-compactness associated to the limit $u \to \infty$ causes long distance divergence on the left hand side. An interesting point is that the short distance, ultraviolet divergence on one side is mapped to the long distance, infrared divergence on the other.

With this caveat understood, the prescription (4.2.1) at least gives a mapping of a CFT to a theory on AdS and vice versa, because formally speaking, a conformal theory is defined by its totality of the correlation functions, and a generally-covariant theory on AdS is specified by the backreaction by the boundary perturbations. The form of the prescription ensures that the theories on both sides share the same symmetry, for example the $SO(4,2)$ symmetry which manifests itself as the conformal symmetry on the right hand side and as the isometry on the left hand side.

In this formal viewpoint, one can surely associate some gravitational theory to the $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory. The Maldacena conjecture then becomes the statement that the gravitational theory thus obtained using the prescription is precisely the type IIB theory on the $\mathrm{AdS}_5 \times S^5$ with flux.

Let us use the prescription above to relate the quantities on both sides. We will establish several such relations in the following.

4.3 Scaling dimension and the mass squared

Our first objective is to find the relation between the scaling dimension of an operator in the CFT and the mass squared of the corresponding field in the AdS. Consider the scalar fields for simplicity. The definition of mass squared itself is subtle in a curved space, since in a curved space a scalar field can have a coupling to the scalar curvature of the form $R\phi^2$, which acts in the same way as the mass term in a spacetime with constant curvature. We choose to define the mass squared m^2 by the following equation

$$(\Box - m^2)\phi = 0 \tag{4.3.1}$$

satisfied by the scalar field where $\Box$ is the ordinary d'Alembertian. We rescale the metric of the AdS space to have curvature radius 1. The original curvature radius can be reinstated easily by the consideration of the dimension of length.

In the Poincaré coordinates (4.1.6), the d'Alembertian becomes

$$\left(\frac{1}{u^3}\frac{\partial}{\partial u}u^5\frac{\partial}{\partial u} + \frac{1}{u}\frac{\partial}{\partial x_i}u\frac{\partial}{\partial x^i} - m^2\right)\phi = 0. \tag{4.3.2}$$

Near the boundary $u \to \infty$ of the AdS space, the second term in the parentheses is subleading. Then the field ϕ has the behavior $\phi \sim u^D$ with $D(D-4) = m^2$. Since the dilatation of the CFT corresponded to the isometry $u \to u/c$, $x^i \to cx^i$, (4.1.7), one finds that the scaling dimension D and the mass squared m^2 have the relation

$$D(D-4) = L^2 m^2 \tag{4.3.3}$$

where we reintroduced the scale L of the AdS space. When solved for D, it reads

$$D = 2 \pm \sqrt{4 + L^2 m^2}. \tag{4.3.4}$$

One notices immediately that for $0 < D < 4$ the mass squared is negative. In a flat spacetime negative mass squared means that the mode is tachyonic and that the system is unstable. In the AdS space it is known however that the instability arises only when

$$L^2 m^2 < -4, \tag{4.3.5}$$

which is called the Breitenlöhner-Freedman bound [41]. Since $D(D-4)$ attains its minimum -4 at $D = 2$, the fields corresponding to the CFT operators never violates the bound.

Another point is that, for negative mass squared which satisfies the bound $-4 \le L^2m^2 \le 0$, there are two solutions $D_\pm$ to $D(D-4) = L^2m^2$. As we saw in section 3.2.1, the unitarity of the CFT requires $D \ge 1$. Thus for $L^2m^2 \ge -3$ there is no ambiguity. However for $-4 \le L^2m^2 \le -3$ there is a genuine ambiguity. Sometimes one can resolve the ambiguity by the requirement of the supersymmetry using the relation $D = 3R/2$ for chiral primary fields as seen in section 3.2.2. There are cases where the ambiguity corresponds to the existence of the renormalization group flow changing the dimension of an operator from D_+ to D_- [42].

4.4 Central charge and the cosmological constant

Let us next see briefly how the central charges of four-dimensional theory is determined from the gravitational theory on AdS. Firstly, we have to retain only the metric to do the calculation, since the central charge is defined by the three-point function of the energy-momentum tensor with itself. For simplicity we suppose the bulk Lagrangian is the pure Einstein-Hilbert form with cosmological constant Λ, without higher derivative terms. An immediate consequence is that the central charge a and c only depends on Λ. The scaling of a and c with respect to Λ is also easy to find. Consider a small perturbation δg_{mn} on a five-dimensional AdS space with curvature radius L, embedded as in (4.1.1) in the six-dimensional space with signature $(4,2)$. Rescale the coordinates X^i by the factor s. Then L, Λ and the on-shell action scale as s, s^{-2} and s^3 respectively, while the perturbation of the metric at the boundary, $\delta g_{\mu\nu}$, remain unchanged. Thus the GKP-W prescription tells us that the central charges scale as s^3. Therefore we obtain

$$a, c \propto \Lambda^{-3/2}. \tag{4.4.1}$$

For the calculation of the precise coefficients using the GKP-W prescription, we refer to the original reference [43]. The result is

$$a = c = \pi^2\Lambda^{-3/2}. \tag{4.4.2}$$

4.5 Triangle anomalies and the Chern-Simons terms

Suppose one has several global internal symmetries Q_I on the CFT side, whose currents are J_I^μ. For the notational simplicity we assume that all the symmetries are Abelian. It can be readily generalized to non-Abelian symmetries.

Under the GKP-W prescription, it implies the existence of the fields A_μ^I in the bulk AdS. The prescription becomes

$$Z_{\text{AdS}}[A_\mu^I\big|_{u=\infty} = \hat{A}_\mu^I] = \langle e^{-\int d^4x \hat{A}_\mu^I J_I^\mu}\rangle. \tag{4.5.1}$$

Firstly, the conservation of currents $\partial_\mu J_I^\mu = 0$ means that the change in $\hat{A}_\mu^I$ by

$$\hat{A}_\mu^I \to \hat{A}_\mu^I + \partial_\mu \chi^I \tag{4.5.2}$$

does not make any difference on the right hand side of (4.5.1). It then implies that the left hand side has the gauge invariance, i.e. A_μ^I are $U(1)$ gauge fields in the bulk.

Secondly, although the currents by definition have no anomaly in the conservation law, they in general have triangle anomalies among them. In a situation when the global symmetry is weakly gauged by the external gauge fields as in (4.5.1), it manifests as the non-invariance of the partition function with respect to the gauges of the external gauge potentials. The form of the non-invariance is severely constrained by the conservation of the currents itself [44], and it has the form

$$\delta_\chi(\langle e^{-\int \hat{A}_\mu^I J_I^\mu}\rangle_{SCFT}) = \int d^4x \frac{1}{24\pi^2}\hat{c}_{IJK}\,\chi^I\, F^J \wedge F^K. \tag{4.5.3}$$

Here the coefficients in front of the right hand side is so chosen that it equals the trace over the labels for Weyl fermions of the cube of the charge matrices, that is,

$$\hat{c}_{IJK} = \operatorname{tr} Q_I Q_J Q_K \tag{4.5.4}$$

for a renormalizable theory. If we normalize the global symmetry so that the fundamental degrees of freedom have integral charges, the coefficients $\hat{c}_{IJK}$ are integers. If phrased in a totally low energy point of view, the condition is that we can couple the system to the external gauge fields with quantized magnetic flux $\int_C F \in 2\pi\mathbb{Z}$.

Returning to the study of the GKP-W prescription (4.5.1), one obtains thus

$$\delta_\chi Z_{\mathrm{AdS}} = \int d^5x \frac{1}{24\pi^2} \hat{c}_{IJK}\, d\chi^I \wedge F^J \wedge F^K \tag{4.5.5}$$

$$= \delta_\chi \left(\int d^5x \frac{1}{24\pi^2} \hat{c}_{IJK}\, A^I \wedge F^J \wedge F^K \right). \tag{4.5.6}$$

Summarizing, the existence of the triangle anomalies implies the existence of the Chern-Simons terms

$$\int d^5x \frac{1}{24\pi^2} \hat{c}_{IJK}\, A^I \wedge F^J \wedge F^K \tag{4.5.7}$$

in AdS_5, which was first pointed out in [3].

There, the comparison of the Chern-Simons term and the triangle anomaly was carried out for the dual pair of the $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills and the type IIB superstrings on $\mathrm{AdS}_5 \times S^5$. The global symmetry on the CFT side is the $SO(6)$ R-symmetry. It should manifest as the $SO(6)$ gauge field in the bulk AdS space and it should have the non-Abelian Chern-Simons term. Indeed, the massless modes of the type IIB superstring theory considered as the theory on AdS_5 form a $\mathcal{N} = 8$ supergravity on AdS_5, whose structure up to two-derivative terms is completely determined by the high degree of supersymmetry [45]. It includes the $SO(6)$ gauge field which minimally couples to the gravitini, and the supersymmetry requires the presence of the non-Abelian Chern-Simons term for $SO(6)$ with the correct magnitude.

The relation of the Chern-Simons coupling in the bulk and the presence of the anomaly in the boundary has also appeared in many places in physics. For example, the quantum Hall fluid has a macroscopic description as the three-dimensional Chern-Simons theory. Thus if we have a droplet of the quantum Hall liquid, the gauge symmetry is apparently violated if analyzed naïvely. The fact is that there is also a chiral edge state on the boundary of the droplet, which is anomalous in itself. Then the anomalies from the boundary of the Chern-Simons theory and from the chiral edge state cancels each other, giving a consistent physical system as a whole. The phenomena has a natural generalization to the higher dimensional solitonic systems [46]. It also manifests in string theory, in which the anomalies from the chiral spectrum on the branes are cancelled by the Chern-Simons coupling of the brane to the bulk [47].

Just as for the triangle anomalies treated in this section, the anomaly among $U(1)$-$T^{\mu\nu}$-$T^{\mu\nu}$ is translated under the GKP-W prescription to the coupling of the form $\int A \wedge \mathrm{tr}\, R \wedge R$. Since this is a higher derivative coupling in the AdS space, it is rather difficult to tackle. In this thesis we mainly assume that the coefficient in front of this term is small compared to the lowest derivative terms, so that the supergravity approximation to the lowest order works.

4.6 Anomalous currents and the Higgs effect

As a final topic in this chapter, we would like to study the supergravity dual for the anomalous global currents in SCFT. Let us first discuss without reference to supersymmetry. The conservation law is modified by the anomaly to be

$$\partial_\mu J^\mu = \mathbb{X} \tag{4.6.1}$$

for a suitable operator $\mathbb{X}$. An example of such current is a chiral $U(1)$ rotation which is broken by the instantons, for which $\mathbb{X} \propto \operatorname{tr} F \wedge F$. Here F is the curvature of the gauge field which is not external, but is the constituent of the CFT considered.

On the gravity dual, we introduce a gauge field A_μ and a scalar ϕ defined on the AdS space and the coupling

$$\int dx^4 (A_\mu J^\mu + \phi \mathbb{X}) \tag{4.6.2}$$

on the boundary. We can see now that if the bulk gauge transformation $A_\mu \to A_\mu + \partial_\mu \epsilon$ is accompanied by the transformation $\phi \to \phi - \epsilon$, the prescription of AdS/CFT correspondence leads to the anomalous conservation law (4.6.1). That the gauge transformation $\delta\phi$ of the field ϕ is nonzero means that the gauge symmetry for A_μ is Higgsed. Summarizing, the anomalous conservation law for a current in the boundary CFT corresponds to the Higgs effect in the bulk AdS [48].

Let us now consider the effect of supersymmetry. Consider an anomalous flavor symmetry. The current J^μ is then incorporated into a current superfield K and the operator $\mathbb{X}$ is the imaginary part of the F-component of the superfield $O \propto \operatorname{tr} W_\alpha W^\alpha$. The supersymmetry completion of (4.6.1) becomes

$$\bar{D}^2 K = O. \tag{4.6.3}$$

This is the celebrated Konishi anomaly [35].

We will see in the next chapter that a current superfield in SCFT corresponds to a vector multiplet and that a chiral multiplet to a hypermultiplet. Thus, the Konishi anomaly is dual in the supergravity description to the Higgsing of a vector multiplet eating a hypermultiplet. After Higgsing, the multiplet is no longer short. Thus the dimension of the operators is not protected anymore. What remains is the relation of the anomalous dimension of J and the anomalous dimension of O [30], which is a consequence of the superconformal symmetry.

Chapter 5

Minimization principle in AdS_5

In this chapter, we review how the a-maximization is translated, under the GKP-W prescription, to the phenomenon in the bulk AdS space. Before doing that, we need to review the structure of the five-dimensional AdS supergravity first. After seeing the supergravity dual of the a-maximization, we will study some application, including the dual of the superconformal deformation, and the dual of the a-maximization with Lagrange multipliers introduced in section 3.6. The content of this chapter is based on the author's paper [12].

5.1 Gauged supergravity in five dimensions

Let us recall the structure of gauged supergravity in five dimensions. The minimum number of supersymmetry generators is eight, and we concentrate on this case. It is called $\mathcal{N} = 2$ supergravity in the supergravity literature. There are several kinds of supermultiplet, and we restrict our attention to the gravity multiplet, vector multiplets and hypermultiplets. We restrict attention to the Abelian vector multiplets for brevity. Non-abelian gauge fields can be incorporated without much extra effort. We follow the conventions used in [49].

5.1.1 Structure of scalar manifolds

Let us first discuss the scalars of the vector multiplets. Let us denote the number of vector multiplets by n_V. Then, there are n_V real scalar fields while there are $n_V + 1$ gauge fields in the theory. When one compactifies one dimension, it will give an $\mathcal{N} = 2$ supergravity theory in four dimensions. As such, the structure of the scalar manifold is determined by a unique function $\mathscr{F}$. A peculiarity in five dimension is that the third derivative of $\mathscr{F}$ governs the Chern-Simons coupling and this fact fixes $\mathscr{F}$ to be cubic,

$$\mathscr{F} = c_{IJK} h^I h^J h^K, \tag{5.1.1}$$

where h^I, ($I = 1, 2, \ldots, n_V + 1$), are the special coordinates. The scalar manifold M_V for the vector multiplet is given by the real n_V-dimensional hypersurface defined by the constraint $\mathscr{F} = 1$ in the space of h^I. These manifolds are known as the very special manifolds[50]. Let ϕ^x, ($x = 1, 2, \ldots, n_V$), parametrize the manifold. It is useful to introduce the following quantities:

$$h_I \equiv c_{IJK} h^J h^K, \qquad g_{xy} \equiv -3 c_{IJK} h^I_{,x} h^J_{,y} h^K, \qquad a_{IJ} \equiv h_I h_J + \frac{3}{2} g^{xy} h_{I,x} h_{J,y}. \tag{5.1.2}$$

We will raise and lower the indices $I, J, \ldots$ by using a_{IJ}. h_I is called the dual special coordinate.

Let us turn to the hypermultiplets. The manifold M_H of the hyperscalars is a quaternionic manifold of real dimension $4n_H$, which means that its holonomy is contained in $Sp(n_H) \times Sp(1)$ [51]. Let q^X, $(X = 1, \dots, 4n_H)$, parametrize the manifold. We will introduce the vielbein f^X_{iA} where $i = 1, 2$ and $A = 1, \dots, 2n_H$ are the indices for $Sp(1)$ and $Sp(n_H)$ respectively. We normalize f^X_{iA} so that $f_{iAX} f^{iA}_Y = g_{XY}$. f^X_{iA} is used to construct the coupling of the gravitino, the hyperino and the hyperscalar. Supersymmetry fixes the $Sp(1)$ part of the curvature [51] so that it is proportional to the triplet of almost complex structures:

$$R_{XYij} = -(f_{XiC} f^C_{Yj} - f_{YiC} f^C_{Xj}). \tag{5.1.3}$$

The relation forces the manifold to be Einstein with a definite proportionality constant[1],

$$R_{XY} = -8n_H(n_H + 2) g_{XY}. \tag{5.1.4}$$

We trade the symmetric combination of two indices $\{ij\}$ for an index $r = 1, 2, 3$ by using the Pauli matrices, that is,

$$T_{ij} = \sigma^r_{ij} T_r \tag{5.1.5}$$

for any tensors. The $Sp(1)$ curvature R^r_{XY} satisfies the relation

$$R^r_{XY} R^s_{XZ} = \frac{1}{4}\delta^{rs} g_{YZ} + \frac{1}{2}\epsilon^{rst} R^t_{YZ}. \tag{5.1.6}$$

The kinetic terms for boson fields are then given by

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{kin,boson}} = -\frac{1}{2}R - \frac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y - \frac{1}{2}g_{XY}\partial_\mu q^X\partial^\mu q^Y \\ - \frac{1}{4}a_{IJ}F^I_{\mu\nu}F^{J\mu\nu} + \frac{1}{6\sqrt{6}}e^{-1}c_{IJK}\epsilon^{\mu\nu\rho\sigma\tau}A^I_\mu F^J_{\nu\rho}F^K_{\sigma\tau} \end{aligned} \tag{5.1.7}$$

where R is the scalar curvature and e is the determinant of the fünfbein.

5.1.2 Gauging and the Potential

We need to introduce a scalar potential to get the AdS$_5$ vacuum. The structure of the potential is extremely restricted by the presence of the high degree of supersymmetries, and it must be accompanied by the gauging of the scalars. It also modifies the supersymmetry transformation.

We will use isometries K^X_I on M_H to covariantize the spacetime derivative

$$\partial_\mu q^X \to \partial_\mu q^X + A^I_\mu K^X_I. \tag{5.1.8}$$

The isometries can be expressed by the relation

$$K^X_I R^{ij}_{XY} = D_Y P^{ij}_I, \tag{5.1.9}$$

using the triplet Killing potential P^{ij}_I. This is required by the consistency of the gauging with the supersymmetry. P^{ij}_I is a generalization of the D-terms for supersymmetric theories with four supercharges. There, the isometry K^i_I of the Kähler manifold should satisfy

$$K^i_I \omega_{i\bar{j}} = \bar{\partial}_j D_I \tag{5.1.10}$$

[1] A peculiarity of the quaternionic manifolds is that the product $Q_1 \times Q_2$ of two quaternionic manifolds is not quaternionic, while $H_1 \times H_2$ of two hyperkähler manifolds is hyperkähler. Indeed, the holonomy group of $Q_1 \times Q_2$ is $Sp(n_H) \times Sp(1) \times Sp(n'_H) \times Sp(1)$, and while $Sp(n_H) \times Sp(n'_H) \subset Sp(n_H + n'_H)$, $Sp(1) \times Sp(1)$is not a subgroup of $Sp(1)$. Moreover, the $Sp(1)$ curvature is fixed in the supergravity as discussed above, and the factor $n_H(n_H + 2)$ does not behave well under the product of two manifolds.

where $\omega_{i\bar{j}}$ is the Kähler form and D_I is the corresponding D-term, or the Killing potential. Hyperkähler and quaternionic manifolds have a triplet of Kähler forms, thus the Killing potential also comes in triplets. For quaternionic manifolds the $Sp(1)$ curvature is nonzero, thus the derivative in the right hand side needs to be covariantized.

The Killing potential P_I^{ij} appears in the Lagrangian. It gives the scalar potential as

$$V = \frac{3}{2}g^{xy}\partial_x P^{ij}\partial_y P_{ij} + \frac{1}{2}g^{XY}D_X P^{ij} D_Y P_{ij} - 2P^{ij}P_{ij} \tag{5.1.11}$$

where $P_{ij} \equiv h^I P_{Iij}$. It also appears in the covariant derivative of the gravitino ψ^i_μ,

$$D_\nu \psi^i_\mu = \partial_\nu \psi^i_\mu + A^I_\nu P^i{}_{jI}\psi^j_\mu + \cdots . \tag{5.1.12}$$

$P^i{}_{jI}$ enters in the covariant derivative of the gaugino as well. It appears also in the supersymmetry transformation laws listed below:

$$\delta_\epsilon \psi^i_\mu = D_\mu \epsilon^i + \frac{i}{\sqrt{6}}\gamma_\mu \epsilon^i P_{ij} + \cdots , \tag{5.1.13}$$

$$\delta_\epsilon \phi^x = \frac{i}{2}\bar{\epsilon}^i \lambda^x_i , \tag{5.1.14}$$

$$\delta_\epsilon \lambda^i_x = -\epsilon_j \sqrt{\frac{2}{3}}\partial_x P^{ij} + \cdots , \tag{5.1.15}$$

$$\delta_\epsilon q^X = -i\bar{\epsilon}_i f^{XiA}\zeta_A , \tag{5.1.16}$$

$$\delta_\epsilon \zeta_A = \frac{\sqrt{6}}{4}\epsilon^i f_{XiA} K^X_I h^I + \cdots . \tag{5.1.17}$$

where λ_x and ζ_A are the gaugino and the hyperino, respectively.

Let us discuss more about the isometry of the hyperscalars. The relation (5.1.9) can be solved to give P in terms of K as follows:

$$2n_H P^{ij}_I = D_X K^Y_I R^{Xij}_Y . \tag{5.1.18}$$

Consider a point on M_H so that $K^X_I = 0$, around which one can expand

$$K^X_I = Q^X_{IY} q^Y + O(q^2). \tag{5.1.19}$$

Comparing with (5.1.8), we see that Q^X_{IY} determines the charge of the hypermultiplets. Then P^{ij}_I at the point is given by

$$P^{ij}_I = \frac{1}{2n_H} Q^X_{IY} R^{Yij}_X . \tag{5.1.20}$$

It means that P^{ij}_I is the $Sp(1)$ part of the charges Q^X_{IY}. Assuming Qs to be rational, P^{ij}_I at the point is also rational.

5.2 AdS dual of a-maximization

We would like to see how the a-maximization is translated under the AdS/CFT duality to the supergravity description. As discussed in section 4.5, the existence of the triangle anomaly for the global internal symmetries implies the existence of the Chern-Simons terms with the strength

$$\frac{1}{24\pi^2}\int \hat{c}_{IJK} A^I \wedge F^J \wedge F^K \tag{5.2.1}$$

where $\hat{c}_{IJK} = \operatorname{tr} Q_I Q_J Q_K$. Comparing with the Lagrangian (5.1.7), we can identify the constants $\hat{c}_{IJK}$ on the CFT side (3.4.6) and the constants c_{IJK} on the AdS side (5.1.7) by the relation:

$$c_{IJK} = \frac{\sqrt{6}}{16\pi^2}\hat{c}_{IJK}. \tag{5.2.2}$$

Just in the same way, the chiral anomaly for the global symmetry–gravity–gravity triangle diagram is reproduced by the coupling

$$\int \hat{c}_I A^I \wedge \operatorname{tr} R \wedge R \tag{5.2.3}$$

where R is the curvature two-form of the metric. It is, however, a higher derivative effect in the AdS side [52] so we neglect them in the rest of the thesis. This means on the CFT side that we restrict attention to theories in which the chiral anomaly concerning gravity is much smaller than the chiral anomaly among three $U(1)$ symmetries.

For the rest of the section let us assume that the hyperscalar[2] is at the point where $K_I^X = 0$ and concentrate on the behavior of the vector multiplets. Let us denote the charges of the hypermultiplet and the Killing potential by Q_{IY}^X and $P_I^{ij} = Q_{IY}^X R_X^{Yij}/(2n_H)$, respectively, as in (5.1.19).

In order for the four-dimensional theory to be superconformal, the five-dimensional bulk should be AdS, and there should be eight covariantly constant spinors. To achieve this, we need to set the gaugini variation (5.1.15) to be zero,

$$P_I^r \langle h^I_{,x} \rangle = 0. \tag{5.2.4}$$

We denoted the value of the quantity at the AdS vacuum by enclosing in the angle brackets as $\langle \cdots \rangle$. This condition says that the three vectors P_I^r, $r = 1, 2, 3$, is perpendicular to the n_V row vectors $\langle h^I_{,x} \rangle$ as vectors with $n_V + 1$ columns, which in turn means that P_I^r are parallel. Thus we can use the $Sp(1)$ global R-symmetry to set $P_I^r = \delta^{3r} P_I$ for some constants P_I. Then the equation (5.2.4) reduces to

$$P_I \langle h^I_{,x} \rangle = 0. \tag{5.2.5}$$

This is an extremization condition for the superpotential $P \equiv P_I h^I$.

Now one can determine the commutation relations among the global symmetries which are respected by the vacuum. They are the isometries of AdS$_5$, eight supercharges, and n_V global $U(1)$ symmetries from the gauge fields A^I_μ. From the covariant derivative of the gravitino (5.1.12), one finds that supercharges have charge $\pm P_I$ under the I-th global $U(1)$. Thus we can identify the quantity P_I introduced above and the quantity $\hat{P}_I$ in the SCFT side which was introduced in (3.4.2). We will not distinguish P_I and $\hat{P}_I$ in the following.

We have found the mapping under AdS/CFT duality of the basic constants $\hat{c}_{IJK}$ and $\hat{P}_I$ in the SCFT and c_{IJK} and P_I in the supergravity. Now we can study how the a-maximization is translated on the gravity side. Let us resume the study of the implication of the condition (5.2.5) and recall the

[2] The global structure of M_H does not concern us. If one wants a concrete example, one can think of M_H as one of the Wolf spaces such as $Sp(n_H, 1)/Sp(n_H)\times Sp(1)$, and think of the Killing vectors as induced by the subgroup of the denominator $Sp(n_H) \times Sp(1)$. However, only the local properties of the metric near the zero of the Killing vectors are relevant, as long as we restrict our attention to the charges and the mass squared of the scalars as we will see below. Moreover, the existence of the tower of Kaluza-Klein excitations means that the quaternionic manifolds for the AdS dual would be infinite dimensional in general.

Another thing one should notice is that the hyperscalar contains the dilaton, when the five-dimensional supergravity arises as the compactification of the type IIB string theory. It means that in general there are corrections to the metric of the hyperscalars. Hence the final metric will not be as simple as the one for the Wolf spaces. The global structure will be important if we study the flow between two supersymmetric vacua [53].

constraint $c_{IJK}h^I h^J h^K = 1$, which implies that $h_I h^I{}_{,x} = c_{IJK}h^I h^J h^K_{,x} = 0$. Thus h_I also is perpendicular to $n_V - 1$ vectors $h^I_{,x}$, from which we deduce that

$$\langle h_I \rangle = c_{IJK}\langle h^J \rangle \langle h^K \rangle \propto P_I. \tag{5.2.6}$$

This is the attractor equation in the five-dimensional gauged supergravity[54].

Let us now identify the superconformal R-symmetry $R_{SC} = r^I Q_I$. From the supersymmetry transformation law for the hypermultiplets (5.1.16) and (5.1.17), we can calculate the anticommutator of the supercharges acting on the hyperscalars. The result is

$$\{\delta_\epsilon, \delta_{\epsilon'}\}q^X = -i\frac{\sqrt{6}}{4}(\bar{\epsilon}\epsilon')\langle h^I \rangle K_I^X + \cdots, \tag{5.2.7}$$

from which we deduce that the anticommutator of the supercharges contains a $U(1)$ rotation $\propto \langle h^I \rangle Q_I$. This $U(1)$ symmetry is identified under the AdS/CFT duality with the $U(1)_R$ symmetry in the superconformal algebra (3.1.18). Thus we find

$$r^I = t\langle h^I \rangle. \tag{5.2.8}$$

where t is some proportionality constant. Let us next fix t. The gauge transformation law for the gravitino (5.1.12) signifies that the superconformal R-charge of the gravitino is $r^I P_I$. Considering that the superconformal R-symmetry is defined to rotate the gravitino by charge one, we need $r^I P_I = 1$. Thus we get

$$r^I = \langle h^I \rangle / \langle P \rangle, \tag{5.2.9}$$

where $\langle P \rangle = \langle h^I \rangle P_I$. Recall that a flavor symmetry $t^I Q_I$ satisfies $P_I t^I = 0$, see (3.4.4). Plugging this into the attractor equation (5.2.6), we obtain

$$c_{IJK}\langle h^J \rangle \langle h^K \rangle t^I \propto P_I t^I = 0. \tag{5.2.10}$$

This is precisely the condition (3.4.11) for theories with no chiral anomaly concerning gravity.

The other equation (3.4.12) is, by using (5.1.2), translated to the positivity of the metric of the scalar manifold [55]. To see this, let us recall that the n_V vectors $\langle h^I_{,x} \rangle$ spans the vector space F defined by the condition

$$F = \{t^I \mid P_I t^I = 0\}. \tag{5.2.11}$$

Thus the positivity of the matrices $-\hat{c}_{IJK}r^I$ acting on F, equation (3.4.12), is translated to the positivity of the matrix $-\hat{c}_{IJK}r^I \langle h^J_{,x} \rangle \langle h^K_{,y} \rangle$, which is precisely the metric (5.1.2) of the vector multiplet scalars.

The maximization of the trial a-function $a(s^I)$ and the extremization of the superpotential $P = P(h^I)$ can be associated more explicitly. Let us generalize the relation (5.2.9) and relate the parameter for the trial R-symmetry $s^I Q_I$ and the value of the special coordinates h^I by the formula $s^I = h^I/(P_J h^J)$. Then, we have

$$a(s) \propto c_{IJK}s^I s^J s^K = \frac{c_{IJK}h^I h^J h^K}{(P_I h^I)^3} = (P_I h^I)^{-3}. \tag{5.2.12}$$

Thus, the trial a-function of the SCFT is precisely the inverse cube of the superpotential. Now it is trivial to see that the minimization of a is the maximization of P !

Let us carry out another consistency check. As was discussed in section 4.4, in a five-dimensional gravitational theory with the action

$$S = \frac{1}{2}\int d^5x\,\sqrt{g}(R + 12\Lambda + \cdots), \tag{5.2.13}$$

the central charge a is given by

$$a = \pi^2 \Lambda^{-3/2}. \tag{5.2.14}$$

At the AdS vacuum, the negative of the vacuum energy is given by the potential (5.1.11) so that

$$6\Lambda = -V = 4(P_I \langle h^I \rangle)^2. \tag{5.2.15}$$

Plugging the relation (5.2.9) and (5.2.15) into (5.2.14), we obtain

$$a = \pi^2 \left(\frac{3}{2}\right)^{3/2} c_{IJK} r^I r^J r^K = \frac{9}{32} \hat{c}_{IJK} r^I r^J r^K. \tag{5.2.16}$$

This agrees with the result from the field theory (3.3.9).

5.2.1 Mass squared of the vector multiplet scalar

The result presented here is a preparation for section 5.2.3 and section 5.3.

We would like to study next the behavior of scalars in the vector multiplet around the vacuum $\langle h^I \rangle$. We can calculate the second derivative of the potential there by using the special geometry relation

$$h^I_{,x;y} = \frac{2}{3} h^I g_{xy} + T_{xyz} h^I_{,w} g^{zw} \tag{5.2.17}$$

where T_{xyz} is a completely symmetric tensor on M_V defined by this equation. Then,

$$D_x \partial_y V \big|_{h^I = \langle h^I \rangle} = -g_{xy} \frac{8}{3} \langle P \rangle^2 \tag{5.2.18}$$

Thus, the mass squared for all the scalar fields is negative with $m^2 = -4\Lambda$. Recall the classic relation (4.3.3),

$$m^2/\Lambda = D(D-4), \tag{5.2.19}$$

between the mass m in the AdS and the dimension D in the CFT. Then we have $D = 2$ for all the n_V scalar fields, which barely saturates the Breitenlöhner–Freedman bound[41] and thus the system is stable [58]. It is also easy to see that they have no R-charges. This is as it should be, because the scalar component of a vector multiplet corresponds to the lowest component of the current superfield, whose dimension is protected to be two and whose R-charge is zero, as seen in sec. 3.2.2.

5.2.2 Dual of Scalar Chiral Primaries in SCFT

An important kind of multiplets in the $d = 4$, $\mathcal{N} = 1$ SCFT is the chiral multiplet, whose lowest component is a complex scalar we denote by $\mathcal{O}$. As seen in section 3.2.2, the dimension D and the R-charge R is related through $D(\mathcal{O}) = 3R(\mathcal{O})/2$. We would like to identify its supergravity dual. A natural candidate will be a hypermultiplet, which comes in quartets of real scalars. Chiral primaries, however, comes in pairs of real scalars. Naïvely there is twice the number of freedom in supergravity. We will see shortly below that the extra two degree of freedom corresponds to the F-component of the superfield $\mathcal{O}$.

Consider n_H hypermultiplets q^X with charges Q^X_{IY} under I-th $U(1)$ symmetry, where $X, Y = 1, \ldots, 4n_H$. The charges under the superconformal R symmetry are given by

$$Q^X_Y \equiv r^I Q^X_{IY} = \langle h^I \rangle Q^X_{IY} / \langle P \rangle. \tag{5.2.20}$$

First, we would like to study the eigenvalues of Q^X_Y. To simplify the calculation, let us combine the $4n_H$ real scalars into $2n_H$ complex scalars by introducing some complex structure J^Y_X so that Q^X_Y is diagonal. The relation (5.1.20) between P^r and Q^X_Y means that J^Y_X is proportional to $R^{Yr}_X P_r$. Let us form $J^\pm$ from the other two complex structures so that

$$[J, J^\pm] = \pm 2J^\pm. \tag{5.2.21}$$

Then we can calculate the commutation relations of Q^X_Y and three J's with the results

$$[J, Q] = 0, \qquad [Q, J^\pm] = \pm 2J^\pm. \tag{5.2.22}$$

This means that the $2n_H$ eigenvectors can be arranged in pairs q_{iA} with charges r_{iA} , ($i = 1, 2$ and $A = 1, \ldots, n_H$), so that $r_{1A} = r_{2A} + 2$. We further abbreviate so that $r_A \equiv r_{1A}$. One can also check that the supersymmetry relates q_{1A} and q_{2A}.

The mass squared m^2_{iA} for the scalar q_{iA} can then be read off from the second derivative of the scalar potential:

$$\begin{aligned} D_X \partial_Y V|_{h^I=\langle h^I\rangle} &= g^{ZW} D_X D_Z P^{ij} D_Y D_W P_{ij} - 4P_{ij} D_X D_Y P^{ij} \\ &= \frac{3}{2} Q^Z_X Q_{YZ} \langle P\rangle^2 - 4\langle P^{ij}\rangle R^{ij}_{XZ} Q^Z_Y. \end{aligned} \tag{5.2.23}$$

Substituting the diagonalized form of the charge matrix, we obtain the masses of the hypermultiplets as follows:

$$m^2_{1A} = \frac{3}{2} r_A (\frac{3}{2} r_A - 4)\Lambda, \tag{5.2.24}$$

$$m^2_{2A} = (\frac{3}{2} r_A + 1)(\frac{3}{2} r_A - 3)\Lambda. \tag{5.2.25}$$

Thus, the scalar which is dual to q_{1A} under AdS/CFT has dimension $3r_A/2$ and R-charge r_A, and the one for q_{2A} has dimension $3r_A/2 + 1$ and R-charge $r_A - 2$. This combination of dimensions and charges are precisely the ones for the lowest component and the F component of a chiral multiplet.

5.2.3 Dual of Marginal Deformations

Let us next discuss the supergravity dual of the exactly marginal deformations in SCFT,

$$S \to S + \int d^4x d^2\theta \tau_i O_i \tag{5.2.26}$$

where the superconformal R-charge of the operators O_i should be two. As remarked in [59], τ_i form a manifold M_c which parametrize the finite deformation. M_c naturally has a complex structure on it, which comes from the fact that the superpotential terms in $d = 4$, $\mathcal{N} = 1$ field theories have natural holomorphic structure.

We should be able to identify M_c in the framework of supergravity. As found in the last subsection, infinitesimal deformations with chiral primaries correspond to the hypermultiplet scalars. That the R-charge of a chiral primary is two means that the mass squared of the corresponding hypermultiplet scalar is 0 and -3, and we saw the deformation $\int d^2\theta \tau_i O_i = \tau_i [O_i]_F$ corresponds to the two real scalars of mass squared zero. This in turn signifies that, when there are n chiral primaries of R-charge two, the supergravity vacuum comes in families with $2n$ real parameters. Let us call it M^c_{sugra}. This should be the supergravity realization of M_c. From the supersymmetry transformation law, we see that

$$M^c_{\text{sugra}} = \{p \in M_H \mid \langle h^I \rangle K_I^X(p) = 0\}. \tag{5.2.27}$$

One thing is not obvious, however. The hypermultiplet scalars form a quaternionic manifold, which is definitely not a complex manifold. It is because the almost complex structures of a quaternionic manifold is not closed, but *covariantly* closed. Fortunately, we can easily check that the submanifold M^c_{sugra} is a Kähler manifold as follows.

First define $K^X \equiv \langle h^I \rangle K_I^X$ and $P^r \equiv \langle h^I \rangle P_I^r$ for brevity. From the property of the Killing potential $D_X P^r = R^r_{XY} K^Y$, P^r is covariantly constant on M^c_{sugra}. In particular, $P \equiv |P^r|$ is a constant parameter. Thus, J_Y^X defined by

$$J_Y^X \equiv R^r_{YZ} g^{XZ} P_r / P \tag{5.2.28}$$

is an almost complex structure. This J_Y^X is covariantly constant with respect to the metric g_{XY} restricted from M_H onto M^c_{sugra}, because every factor in (5.2.28) is covariantly constant. It tells us that the metric on M^c_{sugra} has $U(n)$ holonomy, which means that M^c_{sugra} is Kähler.

5.3 Dual of a-maximization with Lagrange multipliers

As a final exercise in this chapter, let us study the dual of the a-maximization procedure with Lagrange multipliers, discussed in section 3.6. There, symmetries which are not necessarily conserved were introduced and entered in the trial a-function. As seen in section 4.6, it involves massive vector fields in the dual AdS background. Let us examine how the incorporation of these massive vector multiplets affects the supergravity dual of the a-maximization. Consider the chiral operators $O_a = \operatorname{tr} W_a^\alpha W_{a\alpha}$ which yield kinetic terms for the a-th non-Abelian gauge fields, where $a = 1, \ldots, n'_H$ label the factor of the gauge groups. We do not sum over a inside the trace. Define the isometries K_a^X so that, if the Konishi anomaly for the I-th current superfield is given by

$$\bar{D}^2 J_I \propto m_I^a O_a, \tag{5.3.1}$$

the vector field A_μ^I in the five-dimensional supergravity gauges the direction $m_I^a K_a^X$. K_a^X is non-zero at the vacuum.

Other n''_H hypermultiplets which are not Higgsed are also charged under A_μ^I. We denote the Killing vectors for these by $K^X_{(0)I}$. We assume that this can be expanded as

$$K^X_{(0)I} = Q^X_{IY} q^Y + O(q^2) \tag{5.3.2}$$

as before. The total gauging K_I^X appearing in the supergravity Lagrangian is then given by

$$K_I^X = m_I^a K_a^X + K^X_{(0)I}, \tag{5.3.3}$$

and we denote the corresponding Killing potential as

$$P_I^r = m_I^a P_a^r + P^r_{(0)I}. \tag{5.3.4}$$

Let us study the condition for the AdS vacuum, which can be found by inspecting the hyperino and gaugino transformation laws. A convenient parametrization of the hyperscalars near the vacuum is given as follows: near the zero of $K^X_{(0)I}$, let it be linearly dependent on the scalars $q^{\hat{X}}$ where $\hat{X} =$

$1, \ldots, 4n''_H$. We need $4n'_H$ coordinates in addition. n'_H of them are the gauge orbits along K_a^X. We can take P_a^r as the remaining $3n'_H$ of the coordinates. They are guaranteed to form good local coordinates because

$$D_X P_a^{ij} = R_{XY}^{ij} K_a^X \neq 0. \tag{5.3.5}$$

The hyperino transformation law gives the condition

$$\langle h^I \rangle K_I^X = 0. \tag{5.3.6}$$

Its first consequence is that

$$\langle h^I \rangle m_I^a = 0. \tag{5.3.7}$$

Recall the superconformal R-charge is proportional to $\langle h^I \rangle Q_I$, see (5.2.9). This translates in the SCFT language to the fact that anomalous global currents do not participate in the superconformal R-charge. Assuming other linear combination of $\langle h^I \rangle$ is non-zero, we can see that $q^{\hat{X}} = 0$. However, hyperino variation alone does not fix P_a^r.

Next, let us turn to the gaugino variation

$$\langle h^I_{,x} \rangle P_I^{ij} = 0. \tag{5.3.8}$$

Just as before, it says that the vectors with n_V elements $P_I^{r=1,2,3}$ are all parallel to h_I. Global $Sp(1)$ rotation can be used so that P_I^r is nonzero only for $r = 3$. Let us note that, from the relation (5.3.2), $P_{(0)I}^{r=1,2}$ is quadratic in the fields $q^{\hat{X}}$. Combining with the equation (5.3.4), we get $P_a^{r=1,2} = 0$. Thus, the remaining variables are n_V vector multiplet scalars and n'_H coordinates of the hyperscalar, $P_a^{r=3}$. Now define the superpotential to be the gravitino variation

$$P \equiv h^I P_{(0)I} + P_a^{r=3} m_I^a h^I \tag{5.3.9}$$

where the parameter is the vector multiplet scalars h^I with the constraint $c_{IJK} h^I h^J h^K = 1$ and the hypermultiplet scalars $P_a^{r=3}$. Extremization condition for those scalars yields precisely the conditions (5.3.6) and (5.3.8). Thus, the AdS vacua can be found by extremizing the superpotential P with respect to h^I and $P_a^{r=3}$. We can see that the scalars $P_a^{r=3}$ work as the Lagrange multipliers enforcing the condition $h^I m_I^a = 0$. Surprisingly, the Lagrange multipliers are physical fields on the supergravity side!

The way of introducing multipliers in the gauge theory (3.6.3) and in the supergravity (5.3.9) is not exactly the same. We know that however, when we want to extremize a quantity, say $a(x)$, with respect to x in the presence of the constraint $c(x) = 0$, it is immaterial whether we choose $a_1(x, \lambda) \equiv a(x) + \lambda c(x)$ or $a_2(x, \lambda) \equiv f^{-1}(f(a(x)) + \lambda c(x))$. The difference between (3.6.3) and (5.3.9) is of this form, thus of no relevance.

We have seen in section 5.2 that n_V vector multiplet scalars corresponds to the trial R-charge through the relation (5.2.9), and that the dual SCFT operator is the scalar in the current superfield. Then it is natural to ask the same question on the scalars $P_a^{r=3}$. It acted as the Lagrange multipliers in the superpotential extremization. Let us identify the SCFT dual of the scalar $P_a^{r=3}$. The fact that $\langle h^I \rangle P_I^r$ is nonzero only for $r = 3$ means that the superconformal R-symmetry is the $U(1)$ subgroup specified by σ^3 of the global $Sp(1)$ R-symmetry of the ungauged theory. This can be read off just as in the discussion in the section 5.2. This tells us that $P_a^{r=3}$ has zero superconformal R-charge, while $P_a^{r=1,2}$ has charge two.

We have already seen that the gauge orbit K_a^X corresponds to the topological density $\mathrm{tr}\, F^a \wedge F^a$. Three real scalars P_a^r are its supersymmetric partners. From the discussions in section 5.2.2, we can infer that the operators P_a^r correspond under AdS/CFT duality to the three operators

$$\mathrm{tr}\, F^a_{\mu\nu} F^a_{\mu\nu}, \qquad \mathrm{Re}\,\mathrm{tr}\, \lambda^a_\alpha \lambda^{a\alpha}, \quad \text{and} \quad \mathrm{Im}\,\mathrm{tr}\, \lambda^a_\alpha \lambda^{a\alpha}. \tag{5.3.10}$$

Comparing the R-charges, we find that $P_a^{r=1,2}$ are the dual for the gaugino bilinears and that $P_a^{r=3}$ corresponds to the kinetic term $\mathrm{tr}\, F^a_{\mu\nu} F^a_{\mu\nu}$. Let us recall that the prescription of AdS/CFT duality [2, 3] means that there is a boundary interaction

$$\int dx^4 P_a^{r=3}\, \mathrm{tr}\, F^a_{\mu\nu} F^a_{\mu\nu}. \tag{5.3.11}$$

Thus, we found that the Lagrange multiplier $P_a^{r=3}$ corresponds precisely to the gauge coupling constant under AdS/CFT duality. This relation strengthens the identification proposed by Kutasov and reviewed in sec. 3.6 of the Lagrange multiplier and the gauge coupling constant.

Chapter 6

Sasaki-Einstein manifolds

The purpose of this chapter is to introduce the notion of the Sasaki-Einstein manifolds, and discuss its use in string compactification. We also give a detailed review of the geometry of the recently-found $Y^{p,q}$ spaces. We will finish this chapter with the fascinating developments by Martelli, Sparks and Yau [15, 16] which give a method to obtain the volume of the Sasaki-Einstein manifold, given a topological structure without determining the full metric. The references just cited also contain a pedagogical review on the Sasakian and Sasaki-Einstein manifolds.

6.1 Definitions

Consider a manifold X with the metric ds^2_X, and make a manifold $C(X)$ which is one dimension higher than X by adjoining the coordinate r with the metric

$$ds^2_{C(X)} = dr^2 + r^2 ds^2_X. \tag{6.1.1}$$

$C(X)$ is called the (metric) cone of X. For any cone there is a vector field $r\partial_r$ which generates the dilation along the radial direction. We assume hereafter that X is odd dimensional.

Basic definitions are :

- X is contact if $C(X)$ is symplectic,
- X is Sasakian if $C(X)$ is Kähler, and
- X is Sasaki-Einstein if $C(X)$ is Calabi-Yau.

The definition of the Calabi-Yau condition here is that the holonomy should reduce to $SU(n)$ if $C(X)$ is of real dimension $2n$. It is equivalent to imposing the Kähler and the Ricci-flat conditions simultaneously. For compact manifolds, the vanishing of the first Chern class $c_1 = 0$ is not just necessary but also equivalent, thanks to Yau's solution to the Calabi conjecture. In the case of the cones, however, we do not yet have the non-compact version of Yau's theorem. Thus the number of cones which is known to admit a Calabi-Yau metric is quite limited. Still, much of the analysis can be carried out with the knowledge of $c_1 = 0$ and the assumption of the existence of a Calabi-Yau metric. The analysis in sec. 6.4 and 6.5 is done from this perspective.

Let us spell out more explicitly on the conditions. A symplectic manifold of dimension $2r$ has, by definition, a closed non-degenerate two-form ω, and the volume form is given by ω^r. If it is a cone, one can take the interior product with the dilation $r\partial_r$. The outcome is the one-form

$$\eta = \iota_{r\partial_r}\omega. \tag{6.1.2}$$

η is called the contact form and is defined on X. it satisfies $d\eta = \omega$, and thus the volume form of X is given by $\eta \wedge (d\eta)^{r-1}$.

If X is Sasakian, one can use the complex structure J on the cone and the dilation $r\partial_r$ to construct a vector field

$$R = Jr\partial_r \tag{6.1.3}$$

which is an isometry of the base X. It is called the Reeb vector field of the Sasakian structure, and it can also be obtained from the contact form η and the metric g of X. One trivial but important property of the Reeb vector is that it satisfies

$$\langle \eta, R\rangle = 1. \tag{6.1.4}$$

A Sasakian manifold is called regular or quasi-regular if one can divide X by the action of the Reeb vector to form a manifold or an orbifold, respectively. If the Reeb vector cannot be used to divide X to obtain a space with one dimension less, the Sasaki manifold is called an irregular Sasaki manifold.

Another way of expressing it is that X is irregular if the orbit does not close. X is regular if the orbit of the Reeb vector has finite, fixed periodicity X, and it is quasi-regular if the orbit has a fixed periodicity ℓ for generic orbit, but on some special orbits the periodicity drops to ℓ/n for an integer n. Then the special orbit corresponds in the quotient to an n-fold orbifold point. Near the special orbit, a five-dimensional Sasakian manifold has the form

$$\mathbb{C}^2 \times \mathbb{R}/\sim \tag{6.1.5}$$

where the relation $\sim$ is defined by $(\vec{z}, r) \sim (\gamma\vec{z}, r + 2\pi/n)$ for $(\vec{z}, r) \in \mathbb{C}^2 \times \mathbb{R}$ and γ is a linear transformation on $\mathbb{C}^2$ with $\gamma^n = 1$. The isometry acts by the shift of the coordinate r. One can easily check that the generic orbit with $\vec{z} \neq 0$ has periodicity 2π, while that of the special orbit above $\vec{z} = 0$ is $2\pi/n$. The base near $\vec{z} = 0$ has the form $\mathbb{C}^2/\gamma$ with n-th order orbifold point at the origin.

If X is Sasaki-Einstein, we normalize the proportionality constant to be four so that

$$R_{mn} = 4g_{mn} \tag{6.1.6}$$

and we say the volume of the metric in this normalization simply as the volume of the Sasaki-Einstein manifold. A Sasaki-Einstein manifold has a covariantly constant spinor in the sense that

$$(D_m + \gamma_m)\psi = 0 \tag{6.1.7}$$

has an everywhere-nonzero solution. Covariant constancy usually means the condition $D_m\psi = 0$. The presence of an extra γ_m term in the covariant derivative is essential here. The Reeb vector can be constructed from the bilinear of ψ as

$$R^m \propto \bar{\psi}\gamma^m\psi. \tag{6.1.8}$$

The existence of the solution to (6.1.7) follows from the definition of the Sasaki-Einstein property using the Calabi-Yau cone. Indeed, the Calabi-Yau condition of the cone implies the existence of the solution to

$$D_m^{(6)}\psi = 0, \tag{6.1.9}$$

where $D_m^{(6)}$ is the covariant derivative of the cone. Comparing the spin connection on X and that on $C(X)$, one finds $D_m^{(6)} = D_m + \gamma^r\gamma^m$ for m in the direction of the base X. The translation of the Weyl spinor of $SO(6)$ to the spinor of $SO(5)$ and the gamma matrices on them then reveals that (6.1.9) implies (6.1.7).

Locally in a patch, the metric of a Sasaki-Einstein manifold can be put in the canonical form

$$ds_5^2 = (d\psi + \sigma)^2 + ds_4^2 \tag{6.1.10}$$

where ∂_ψ is the Reeb vector and ds_4^2 is the metric of the Kähler-Einstein base B with the normalization

$$R_{i\bar{j}} = 6g_{i\bar{j}}, \tag{6.1.11}$$

and the connection σ satisfies

$$d\sigma = 2\omega_4 \tag{6.1.12}$$

where ω_4 is the Kähler form of B. If X is regular or quasi-regular, the base B of each patch of X can be glued consistently to give X the structure of the S^1 bundle over a manifold or an orbifold B, while it is impossible if X is irregular.

6.2 Use in string compactification

Let us consider how we can use five-dimensional Sasaki-Einstein manifolds X in string compactification. Firstly, one can use the metric cone $C(X)$ in supersymmetric compactification of type II superstrings. Indeed, on the spacetime of the form Minkowski $\times\, C(X)$ there exist eight unbroken supercharges, because $C(X)$ is Calabi-Yau. For the type IIB superstring, one can introduce D3-branes which fill the four-dimensional Minkowski spacetime and are pointlike in $C(X)$. If we place them on a generic point on the cone, it will break the supersymmetry in half. The most interesting situation is when all of them are on the tip of the cone. Since the generic metric cone $C(X)$ is not an orbifold, we cannot carry out a perturbative analysis of the open string spectrum and the remaining supersymmetry on it, but the deformation from the orbifold to the cone over generic Sasaki-Einstein manifolds suggests it preserves four supercharges.

There is a supergravity solution which represents the backreaction of the D3-branes on the tip, which was discussed in section 2.4. Taking the near horizon limit, we get a supergravity solution of the form $\mathrm{AdS}_5 \times X_5$, with N units of self-dual five-form flux through X_5. The five-form flux is proportional to the volume form, that is,

$$F_{(5)} = \frac{2\pi N}{\mathrm{Vol}(X_5)}\left(\mathrm{vol}(\mathrm{AdS}) + \mathrm{vol}(X_5)\right) \tag{6.2.1}$$

where, as always, Vol means the integrated volume and vol means the volume form in the conventional normalization (6.1.6). The isometry enhances from the Poincaré symmetry to the conformal symmetry, thus the number of unbroken supercharges should double.

Let us check explicitly that there are 8 unbroken supercharges. The supersymmetry transformation law for the gravitino is given by

$$\delta\psi^i_\mu = (D_\mu + \gamma_\mu \not{F})\epsilon^i \tag{6.2.2}$$

where $\not{F}$ is the contraction of F with the gamma matrices in ten dimensions, which makes any totally antisymmetric tensor into a bispinor, and ϵ^i are two Majorana-Weyl spinors. The unbroken supersymmetry is the one for which the right hand side is zero. Weyl spinors of $SO(9,1)$ decomposes under its subgroup $SO(4,1)\times SO(5)$ as the product of respective spinors, and the Majorana condition comes from the combination of the pseudo-reality conditions on the $SO(4,1)$ and $SO(5)$ spinors. Then the equation reduces, in the background we are considering, to the decoupled equations

$$(D_\mu + \gamma_\mu)\epsilon_{\mathrm{AdS}} = 0, \tag{6.2.3}$$

$$(D_m + \gamma_m)\epsilon_X = 0. \tag{6.2.4}$$

The original solution is given by a suitable combination of $\epsilon_{\mathrm{AdS}} \otimes \epsilon_X$, with the reality condition imposed. (6.2.3) has four complex solution, while (6.2.4) is precisely the covariant constancy (6.1.7), which has one complex solution. By taking into account the reality condition and two supertranslation of the type IIB supergravity, we obtain eight real supercharges in total.

The analysis above makes clear that the curvature radii of AdS_5 and X_5 need to be the same in order to have a supersymmetric background. One can also easily understand that what determines the metric curvature is the field strength F_5 or the number of flux per unit volume, which is reflected in the formula (6.2.6) below.

Presence of a covariantly constant spinor with $D_m\psi = 0$ means the reduction of the holonomy of the manifold considered. The additional term γ_m makes the situation different and the holonomy of X in the usual sense does not reduce. Thus the contribution from the RR-flux in the transformation law (6.2.2) is crucial to the existence of the unbroken supersymmetry.

Finally, let us see how the central charge of the SCFT is related to the geometry of the Sasaki-Einstein manifold X. Suppose there are N units of the five-form flux through X with volume V. The equation of motion in ten dimensions is

$$R_{\mu\nu} = \frac{c}{24} F_{\mu\alpha\beta\gamma\delta} F_{\nu}{}^{\alpha\beta\gamma\delta} \tag{6.2.5}$$

with $c = 16\pi^6\alpha'^4 g_s^2$ in the convention we set up in section 2.1. Then the curvature radius of the AdS space is found to be

$$L^8 = \frac{c}{4}\left(\frac{2\pi N}{\mathrm{Vol}\, X}\right)^2. \tag{6.2.6}$$

Thus, the five-dimensional action for the metric is

$$\frac{L^5\,\mathrm{Vol}\, X}{8c\pi} \int \sqrt{g}(R + 12L^{-2}). \tag{6.2.7}$$

We need to use the Weyl rescaling of the metric to bring the Lagrangian above to the Einstein frame in five dimensions defined by

$$S = \frac{1}{2}\int \sqrt{g}(R + 12\Lambda). \tag{6.2.8}$$

After the Weyl transformation, one finds the cosmological constant Λ to be

$$\Lambda = \left(\frac{4\,\mathrm{Vol}\, X}{N^2\pi}\right)^{2/3}. \tag{6.2.9}$$

By combining with (4.4.2) of the relation of Λ and a, we obtain

$$a = c = \frac{N^2\pi^3}{4\,\mathrm{Vol}\, X}. \tag{6.2.10}$$

Before moving on to the explicit examples, we would like to consider the parity symmetry of these theories. Since the type IIB supergravity which we took as the starting point is chiral, it does not have the parity symmetry, while the sign flip of even number of coordinates is a symmetry. Furthermore, the type IIB supergravity has a discrete symmetry Ω called the worldsheet parity, which can be used to obtain the type I supergravity. Thus, if we flip three coordinates of the AdS space and odd number of internal coordinates of the Sasaki-Einstein space and then perform the operation Ω, it is a symmetry of the resulting compactification. When the point of view of the boundary CFT is taken, it corresponds to the parity flip combined with the reversal of one of the global symmetry charges. Thus, the CFT corresponding to these compactification in general is invariant under CP. It implies that the corresponding CFT does not have the extra central charge discussed in the footnote in p. 19.

6.3 Examples

6.3.1 S^5

The easiest five-dimensional Sasaki-Einstein manifold is the round five-sphere S^5. The metric cone $C(S^5)$ over it is the flat six-dimensional space $\mathbb{R}^6$. The unit sphere with radius one satisfies $R_{mn} = 4g_{mn}$, and the volume is π^3.

It is known that one can introduce many other quasi-regular Sasaki-Einstein structure on the five-manifold which is topologically S^5 [60]. The cone over it has the complex structure of the Brieskorn-Pham singularity $Y_{a,b,c,d}$

$$x^a + y^b + z^c + w^d = 0 \tag{6.3.1}$$

with integers a, b, c and d. Sliced at $|x|^2 + |y|^2 + |z|^2 + |w|^2 = 1$, it yields a five-dimensional manifold $L_{a,b,c,d}$. For suitable choices of four integers a, b, c and d, it is topologically an S^5 [1]. For a suitable combination of a, b, c and d, one can show that on the quotient of $Y_{a,b,c,d}$ by the action $\mathbb{C}^\times$ there exists a (possibly orbifold) Kähler-Einstein metric, using a generalization of the Yau's theorem. The authors of [60] thereby showed that at least 68 choices of a, b, c and d yield inequivalent Sasaki-Einstein structures on S^5. They can be distinguished by the action of the Reeb vector on S^5, i.e. they differ as G-manifolds with $G = U(1)$. The volume of such spaces were calculated before the work [60] by the string theorists in [61], assuming the existence of a Sasaki-Einstein metric.

6.3.2 $T^{1,1}$

Let us consider the homogeneous manifolds

$$T^{p,q} = SU(2) \times SU(2)/U(1)_{p,q} \tag{6.3.2}$$

where $U(1)_{p,q}$ is the subgroup generated by $p\sigma^3_{(1)} + q\sigma^3_{(2)}$. It is an S^1 bundle over $S^2 \times S^2$, where the Chern classes over the two S^2 are p and q, respectively. It is known that one can introduce an Einstein metric on any of these spaces by rescaling the length of the S^1 fiber and the radii of two two-spheres. Of these, only $T^{1,1}$ is Sasaki-Einstein. It has the topology of $S^2 \times S^3$, and the metric is

$$ds^2 = \frac{1}{6}\sum_{i=1,2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2) + \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2. \tag{6.3.3}$$

The metric is already in the canonical form (6.1.10). Thus the Reeb vector is $3\partial_\psi$, and $3d(\cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)$ is twice the Kähler form of the base $S^2 \times S^2$. The Sasaki-Einstein structure is regular.

The volume of $T^{1,1}$ was first compared to the central charge in the reference [56] by using the formula (6.2.10). Indeed, the volume is

$$\mathrm{Vol}(T^{1,1}) = \frac{16}{27}\pi^3 \tag{6.3.4}$$

while the corresponding quiver theory we introduced in section 2.4.2 has four bifundamental chiral superfields with $R = \frac{1}{2}$ and two adjoint gaugini with $R = 1$ so that

$$a = \frac{3}{32}(3\,\mathrm{tr}\,R^3 - 5\,\mathrm{tr}\,R) = \frac{27}{32}N^2. \tag{6.3.5}$$

These two quantities satisfy the relation (6.2.10), giving another non-trivial check of the duality.

[1] If one carries out similar construction with more complex variables, one gets many exotic spheres.

6.3.3 $Y^{p,q}$

For a long time, S^5 and $T^{1,1}$ we just described were the only known examples of smooth five-dimensional Sasaki-Einstein manifold with explicit metric. The situation changed completely when the paper [13] appeared. The authors constructed a countably infinite number of inequivalent Sasaki-Einstein metrics on $S^2 \times S^3$. In this subsection we review some of their salient features.

Consider the following metric parametrized by a real number a with coordinates α, θ, ϕ, y and ψ:

$$ds^2 = ds_B^2 + w(y)(d\alpha + A)^2, \tag{6.3.6}$$

$$ds_B^2 = \frac{1-y}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{dy^2}{6p(y)} + \frac{q(y)}{9}(d\psi - \cos\theta d\phi)^2, \tag{6.3.7}$$

$$A = f(y)(d\psi - \cos\theta d\phi) \tag{6.3.8}$$

where

$$w(y) = 2\frac{a-y^2}{1-y}, \quad p(y) = \frac{w(y)q(y)}{6}, \quad q(y) = \frac{a-3y^2+2y^3}{a-y^2}, \quad f(y) = \frac{a-2y+y^2}{6(a-y^2)}. \tag{6.3.9}$$

One can check that it satisfies $R_{mn} = 4g_{mn}$ for arbitrary a. Furthermore, it is Sasaki-Einstein with

$$3\frac{\partial}{\partial\psi} - \frac{1}{2}\frac{\partial}{\partial\alpha} \tag{6.3.10}$$

as the Reeb vector. Thus, if the metric becomes smooth for a suitable range of the variables and the parameter a, we obtain a smooth Sasaki-Einstein manifold.

Firstly let us take $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. Then (θ, ϕ) parametrize a round S^2. Secondly, choose $0 < a < 1$ so that the numerator of $q(y)$, $a - 3y^2 + 2y^3$, has three real roots $y_{1,2,3}$ with $y_1 < 0 < y_2 < 1 < y_3$. Let us then take the range of y to be $y_1 \le y \le y_2$. One can check that

$$a - y^2 > 0, \qquad 1 - y > 0, \qquad a - 2y + y^2 > 0 \tag{6.3.11}$$

under the same assumption.

Now the metric of B_4 is positive definite for $y_1 < y < y_2$. Near the boundary $y \sim y_i$, we can approximate

$$q(y) \sim (y - y_i)q'(y_i). \tag{6.3.12}$$

If we define

$$R = \sqrt{4(y-y_i)/wq'(y_i)} \tag{6.3.13}$$

then the metric near the boundary becomes

$$ds^2 = dR^2 + R^2(d\psi - \cos\theta d\phi)^2. \tag{6.3.14}$$

Thus, taking the range of ψ to be $0 \le \psi \le 2\pi$, one obtains another S^2 parametrized by (y, ψ). The second S^2 is not completely round, and has only $U(1)$ isometry which rotates the angular coordinate ψ. Furthermore, the term $\cos\theta\, d\phi$ which is added to $d\psi$ in the metric of the base B means that the second S^2 is non-trivially fibered over the first S^2. Indeed, one can introduce a natural complex structure on B which makes $B \simeq \mathbb{F}_2$, one of the Hirzebruch surfaces.

The final thing to be done is to ensure that the fibering of α is smooth. Take the periodicity to be $0 \le \alpha \le 2\pi\ell$. Then we have four three-cycles D_1, D_2, S_1 and S_2 at $\theta = 0$, $\theta = 2\pi$, $y = y_1$ and

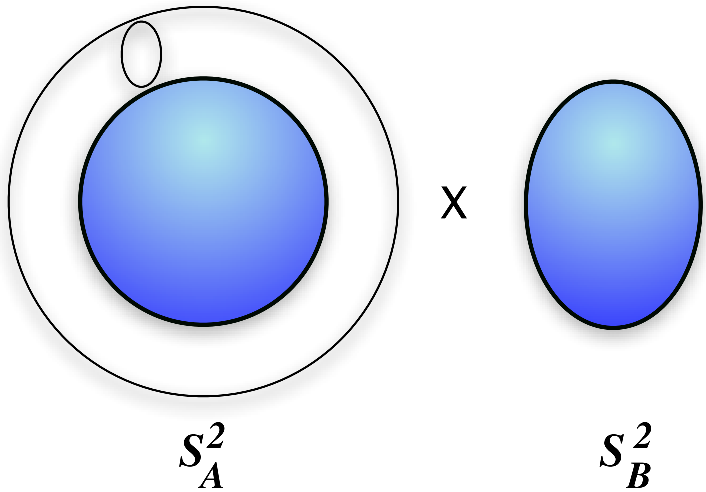

Figure 6.1: Schematic structure of the space $Y^{p,q}$. It is an S^1 bundle over $S^2_1 \times S^2_2$, where the first S^2 is completely round with $SU(2)$ symmetry while the second S^2 has the symmetry only under the $U(1)$ rotation of the axis.

$y = y_2$, respectively. One can show that homologically they can be expressed by the combination of two cycles C_1 and C_2 in the form

$$S_1 \sim C_1 + C_2, \qquad S_2 \sim C_2 - C_1, \qquad D_1 \sim D_2 \sim C_1. \tag{6.3.15}$$

C_1 is the class of the fiber of $\mathbb{F}_2$. The S^1 fiber over $\mathbb{F}_2$ is genuine if and only if its Chern numbers are integers, that is,

$$\int_{C_1} dA = 2\pi\ell p, \quad \text{and} \quad \int_{C_2} dA = 2\pi\ell q \tag{6.3.16}$$

for suitable integers p and q.

One can easily calculate that

$$\frac{1}{2\pi}\int_{S_i} dA = \frac{y_i - 1}{3y_i}. \tag{6.3.17}$$

Thus,

$$\lambda \equiv y_2 - y_1 = \frac{3q}{2p} \tag{6.3.18}$$

needs to be rational. Conversely, if $\lambda \in \mathbb{Q}$ one can solve for $y_{1,2,3}$ and determine the parameter a in the metric. Then we have

$$\ell = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}. \tag{6.3.19}$$

The condition $y_1 < 0 < y_2 < 1 < y_3$ is satisfied if and only if $0 < q < p$.

Summarizing, we obtained a Sasaki-Einstein metric parametrized by two positive integers p and q with $p > q$, whose schematic structure is depicted in figure 6.1. They are called the $Y^{p,q}$ spaces. The coordinate with period 2π is $\gamma \equiv \alpha/\ell$ rather than α. Then the Reeb vector (6.3.10) becomes

$$R = 3\frac{\partial}{\partial\psi} - \frac{1}{2\ell}\frac{\partial}{\partial\gamma}. \tag{6.3.20}$$

ℓ is irrational for generic p and q, thus the Reeb vector field does not close for such a choice of integers. It rather fills the torus parametrized by (ψ,γ). Thus $Y^{p,q}$ are irregular[2]. The construction of these spaces is miraculous, if one recalls that the construction of explicit even-dimensional Kähler-Einstein metric is extremely hard. Indeed, no one knows the explicit metric for, say, a smooth K3.

Before moving to the next topic, let us calculate the volume of $Y^{p,q}$. From the metric, the volume form is

$$\frac{1-y}{6}d\theta\wedge\sin\theta d\phi\wedge\frac{dy}{\sqrt{6p(y)}}\wedge\frac{\sqrt{q(y)}}{3}(d\psi-\cos\theta d\phi)\wedge\sqrt{w(y)}(d\alpha+A)$$
$$=\frac{1-y}{18}d\theta\wedge\sin\theta d\phi\wedge dy\wedge d\psi\wedge d\alpha. \quad (6.3.21)$$

Thus, the volume is easily found to be

$$\mathrm{Vol}(Y^{p,q})=(4\pi)(2\pi)(2\pi\ell)\int_{y_1}^{y_2}\frac{(1-y)dy}{18}=\frac{q^2(2p+\sqrt{4p^2-3q^2})}{3p^2(3q^2-2p^2+p\sqrt{4p^2-3q^2})}\pi^3. \quad (6.3.22)$$

Another comment to be made to the $Y^{p,q}$ spaces is that $Y^{p,q}$, $T^{1,1}$ and the round S^5 exhaust the list of five-dimensional Sasaki-Einstein metrics of cohomogeneity one [62]. Here, the cohomogeneity of a G-manifold is defined to be the codimension of a generic orbit of G. Thus, any other smooth Sasaki-Einstein five-manifolds will be of cohomogeneity at least two.

6.4 Toric Sasaki-Einstein manifolds

S^5, $T^{1,1}$ and $Y^{p,q}$ have large isometry groups. They are $SO(6)$, $SU(2)^2\times U(1)$, and $SU(2)\times U(1)^2$, respectively. At the other extreme, the Sasaki structure ensures the existence of at least one $U(1)$ symmetry, which is generated by the Reeb vector. For example, extra Sasaki-Einstein structures on S^5 briefly discussed in section 6.3.1 have exactly one $U(1)$ isometry. Here we would like to consider Sasaki-Einstein manifolds with $U(1)^3$ isometry[3]. Such manifolds are called toric Sasaki-Einstein, because the cone over it are toric Calabi-Yau cones. A good introduction to complex toric manifolds in general can be found in [63]. A toric Calabi-Yau cone corresponds to an affine toric varieties whose defining vectors lie on a plane, so it has much simpler properties than that of generic toric manifolds.

First, let us define the moment μ_k for an isometry k on a contact manifold by the formula

$$\mu_k=\langle\eta,k\rangle. \quad (6.4.1)$$

It satisfies the usual relation for the symplectic manifold

$$d\mu_k=\iota_k\omega \quad (6.4.2)$$

on the cone, which can be checked using the relation $d\eta=\omega$. μ_k is constant on a orbit of k.

Let the generators of $U(1)^3$ be $X_{1,2,3}$. Then, $\mu_i\equiv\mu_{X_i}$ determines a map

$$\mu: X\to\mathbb{R}^3, \quad x\mapsto(\mu_1(x),\mu_2(x),\mu_3(x)) \quad (6.4.3)$$

[2]Parenthetically, it was even conjectured in the mathematics literature that irregular Sasaki-Einstein metric did not exist. The construction of $Y^{p,q}$ spaces disproved the conjecture explicitly.

[3] Sasaki-Einstein 5-manifolds with exactly $U(1)^2$ isometry can be constructed by dividing $Y^{p,q}$ spaces by a subgroup Γ of their $SU(2)$ isometry of type D or E.

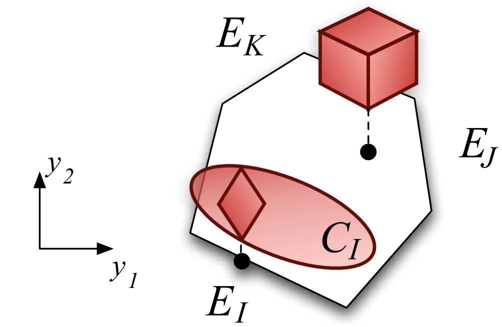

Figure 6.2: Schematic description of a toric Sasaki-Einstein manifold.

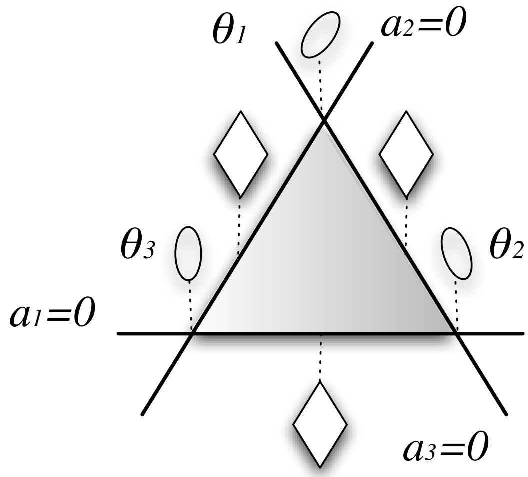

Figure 6.3: S^5 as a toric Sasaki-Einstein manifold.

which is called the moment map. Since the Reeb vector field R satisfies $\langle\eta, R\rangle = 1$, the image lies on a plane in $\mathbb{R}^3$. Furthermore, it is known that the image of the map is a convex polygon P. The inverse image of a point inside P is T^3, while the inverse image of a point on the edge is T^2. Each edge I is labeled by the vector $k_I = k_I^i X_i$ degenerating there, where the three torus T^3 shrink to T^2, see fig. 6.2. We say a toric Sasaki-Einstein manifold to have d edges if the image of the moment map is a d-gon. We normalize k_I for a smooth Sasaki-Einstein manifold so that the metric transverse to the inverse image of the I-th edge is given by

$$dr^2 + r^2 d\theta_I^2 \qquad \text{(no sum on } I\text{)}, \tag{6.4.4}$$

so that $k_I = \partial/\partial\theta_I$. In other words, k_I should generate the standard angular rotation for the transverse direction to the edge. It is known that the Calabi-Yau condition forces k_I to have

$$\langle\eta, k_I\rangle = \frac{1}{3}. \tag{6.4.5}$$

The explanation above was slightly abstract, so we give the toric structure of the round sphere S^5 as an example to clarify the process. The isometry $SO(6)$ includes $U(1)^3$ as the Cartan subgroup, so S^5 is indeed a toric Sasaki-Einstein manifold. Let S^5 be parametrized by three complex variables $z_{1,2,3}$ with the constraint

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1. \tag{6.4.6}$$

The three rotations act by rotating z_i independently. Change the parametrization by defining

$$z_i = \sqrt{a_i} e^{i\theta_i}. \tag{6.4.7}$$

Then S^5 corresponds to the region

$$a_1 + a_2 + a_3 = 1, \qquad a_1, a_2, a_3 \geq 0; \qquad 0 \leq \theta_{1,2,3} \leq 2\pi. \tag{6.4.8}$$

One can show that a_i are precisely the moments of the $U(1)$ actions. The situation is depicted in fig. (6.3).

$Y^{p,q}$ has $SU(2)\times U(1)^2$ as the isometry group, which has $U(1)^3$ as a subgroup. Thus it is also a toric Sasaki-Einstein manifold. We use the conventions introduced in section 6.3.3. Integrally normalized generators of $U(1)$ symmetries are ∂_ψ, ∂_ϕ and ∂_γ. The cycles $D_{1,2}$ and $S_{1,2}$ map to the edges under the moment map, and the vectors

$$k_1 = \partial_\psi + \frac{p-q}{2}\partial_\gamma, \qquad k_2 = \partial_\phi + \partial_\psi, \qquad k_3 = \partial_\psi - \frac{p+q}{2}\partial_\gamma, \qquad k_4 = \partial_\psi - \partial_\phi \tag{6.4.9}$$

degenerate at the edges S_2, D_1, S_1, D_2 respectively. By a change of basis $\partial_{\psi,\phi,\gamma}$ by $SL(3,\mathbb{Z})$,

$$u = \partial_\psi + \frac{p-q}{2}\partial_\gamma, \qquad v = \frac{p-q}{2}\partial_\gamma - \partial_\phi \qquad w = -\partial_\gamma, \tag{6.4.10}$$

k_I is brought to the configuration depicted in figure 6.4[4],

$$k_1 = u, \qquad k_2 = u - v, \qquad k_3 = u + pw, \qquad k_4 = u + v + (p-q)w. \tag{6.4.11}$$

The diagram like figure 6.4 which summarizes the vectors which degenerate at the edges are called the toric diagram of the toric Sasaki-Einstein manifold, while the vectors k_I which specifies the topological structure of a toric Sasaki-Einstein manifold are called the generators of the toric cone, or more plainly as the toric data. The Reeb vector (6.3.20) is, in this basis,

$$R = 3u + \left(\frac{3p-3q}{2} + \frac{1}{2\ell}\right) w. \tag{6.4.12}$$

Suitable orbifolds of the form $S^5/(\mathbb{Z}_2 \times \mathbb{Z}_p)$ and $T^{1,1}/\mathbb{Z}_p$ have the same toric diagram as in figure 6.4 with $q = p$ and $q = 0$ respectively. Hence they are sometimes treated as members of the family $Y^{p,q}$, namely $Y^{p,p}$ and $Y^{p,0}$ when one emphasizes the toric Sasaki-Einstein structure. One needs to be careful however, that the metric of $Y^{p,p}$ and $Y^{p,0}$ are not genuinely of the form presented in section 6.3.3. One has to take suitable rescaling limit to obtain the metric of S^5 or $T^{1,1}$.

In the reference [64] the authors constructed another countably-infinite family of Sasaki-Einstein metric called $L^{a,b,c}$ with precisely $U(1)^3$ isometry, not larger. When combined with $Y^{p,q}$ spaces, they comprises a complete determination of explicit metrics for toric Sasaki-Einstein manifolds with quadrangle as the image of the moment map. Thus, it is natural to endeavor to obtain an explicit metric for toric Sasaki-Einstein manifolds which have more than four edges, although nothing is found yet.

[4] These are precisely the same toric Calabi-Yau cone as the one often used in producing $SU(p)$ gauge theory by means of the geometric engineering in type IIA string theory, and is also extensively studied from that perspective. We are using the same space in type IIB compactification with branes.

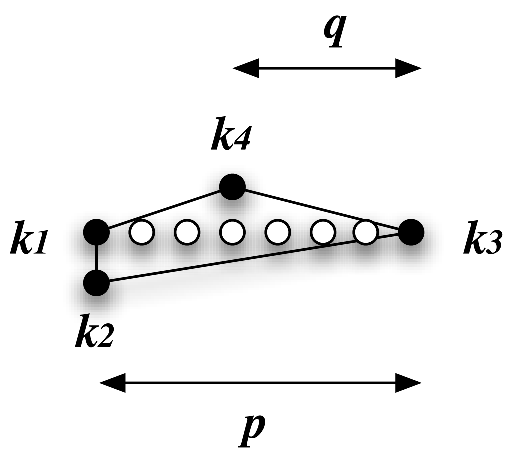

Figure 6.4: Toric diagram for $Y^{p,q}$.

6.5 Volume minimization

Unfortunately, the examples above, S^5, $T^{1,1}$, $Y^{p,q}$ and $L^{a,b,c}$, make up of the complete list of the explicit metrics of the smooth Sasaki-Einstein manifolds constructed so far. It is known, however, through the works by Martelli, Sparks and Yau [15, 16] , one can determine the volume of these manifolds by minimizing a certain functional, without knowing the explicit metric. In this subsection we would like to review these astonishing developments.

6.5.1 Volume and the Einstein-Hilbert action

Let us consider a problem of finding the Sasaki-Einstein metric, given the Sasaki structure on X. The Einstein equation $R_{mn} = 4g_{mn}$ follows from the variational principle starting from the Einstein-Hilbert action,

$$S = \int_X d^5x\,\sqrt{g}(R_X - 12). \tag{6.5.1}$$

Firstly we show that $S = 8\,\mathrm{Vol}(X)$ if the metric g is Sasaki and the cone is topologically Calabi-Yau, where the topological Calabi-Yau condition is the existence of a non-zero holomorphic three-form Ω. The scalar curvature of the cone $Y = C(X)$ is given by

$$R_Y = \frac{1}{r^2}(R_X - 20), \tag{6.5.2}$$

while from the Kähler structure of the cone, one has

$$R_Y = g^{\bar{j}i}\bar{\partial}_{\bar{j}}\partial_i \log\det g_{k\bar{l}}. \tag{6.5.3}$$

$\log\det g_{k\bar{l}}$ is not a good scalar quantity on Y, but we can replace it with the ratio e^{-f} in

$$(g_{i\bar{j}}dz^i d\bar{z}^{\bar{j}})^3 = e^{-f}\Omega\wedge\bar{\Omega}, \tag{6.5.4}$$

because the holomorphy of Ω implies that they drop out when one applies $\bar{\partial}_{\bar{j}}\partial_i$ on it. Thus we have

$$R_Y = \Delta_Y f. \tag{6.5.5}$$

From (6.5.4) one sees that f is independent of r. Thus,

$$\int_{r=0}^{1} d^6x\,\sqrt{g_Y}R_Y = \int_{r=0}^{1} d^6x\,\sqrt{g_Y}\Delta_Y f = 0. \tag{6.5.6}$$

It implies, from (6.5.2), that

$$\int_X d^5x\,\sqrt{g}R_X = 20\int_X d^5x\,\sqrt{g} = 20\,\mathrm{Vol}(X). \tag{6.5.7}$$

Thus the Einstein-Hilbert action becomes

$$S = 8\,\mathrm{Vol}(X), \tag{6.5.8}$$

which was to be shown.

6.5.2 Volume as the equivariant integral

The next trick is to reformulate the volume as the equivariant integral. It starts with the trivial observation that Vol(X) can be written as

$$\mathrm{Vol}(X) = \frac{1}{8}\int_{C(X)} d^6x\,\sqrt{g}e^{-r^2/2}. \tag{6.5.9}$$

When one rewrite this in the form

$$\mathrm{Vol}(X) = \frac{1}{8}\int_{C(X)} e^{-r^2/2}\frac{\omega^3}{3!} \tag{6.5.10}$$

and notice that $H = r^2/2$ is precisely the Hamiltonian for the Reeb vector field for the symplectic form ω, one realizes that it is the equivariant integral of the exponential of the equivariantly closed form $\omega + H$ over the cone. Thus, the volume Vol(X) depends on the Reeb vector alone.

It can be done by utilizing the Duistermaat-Heckman formula, which localizes the integral to the fixed point of the vector field. In our case, the only fixed point is the tip of the cone, which is too singular to determine easily its contribution. In the case of toric Sasaki manifolds, the singularity at the tip can be blown-up to make it smooth, preserving the toric Käher condition. Since it is known that the equivariant integration is conserved by the blowup, one can carry out the localization in a smooth space, and we get the formula for the equivariant integral

$$Z(R) \equiv \frac{1}{8}\int_{C(X)} \frac{\omega^3}{3!}e^{-H_R} \tag{6.5.11}$$

as

$$Z(R) = \frac{\pi^3}{3\langle\eta, R\rangle}\sum_I \frac{\det(k_{I-1}, k_I, k_{I+1})}{\det(R, k_{I-1}, k_I)\det(R, k_I, k_{I+1})}, \tag{6.5.12}$$

where k_I is the I-th toric datum, that is the vector degenerating at the I-th edge, and H_R is the Hamiltonian for the isometry R, which is given by

$$H_R = \langle\eta, R\rangle\frac{r^2}{2}. \tag{6.5.13}$$

Detailed derivation of the formula above can be found in [15], where a slightly different, more direct approach was taken.

As stated in the previous paragraph, $Z(R)$ equals the volume of the base X with the Reeb vector R, if R is normalized so that $\langle \eta, R\rangle = 1$. It is a nice exercise to check that it correctly reproduces the volume of S^5 or $T^{1,1}$. Indeed, for S^5

$$k_1 = (1,0,0), \quad k_2 = (0,1,0), \quad k_3 = (0,0,1); \tag{6.5.14}$$

$$\eta = (1/3,1/3,1/3); \qquad R = (1,1,1), \tag{6.5.15}$$

which yields $Z(R) = \pi^3$, while for $T^{1,1}$

$$k_1 = (1,0,0), \quad k_2 = (1,1,0), \quad k_3 = (1,1,1), \quad k_4 = (1,0,1); \tag{6.5.16}$$

$$\eta = (1/3,0,0); \quad R = (3,3/2,3/2), \tag{6.5.17}$$

which yields $Z(R) = 16\pi^3/27$.

By combining with the result obtained in the previous subsection, we find that the Reeb vector for the Sasaki-Einstein manifold can be found by minimizing the volume, which can be calculated by the equivariant localization, with the explicit formula (6.5.12) for toric Sasaki-Einstein manifolds. Since $\langle \eta, R\rangle = 1$ by definition, the number of the indeterminates are two. The volume is often termed $Z(R)$ as the function of the Reeb vector, so that the method above is called the Z-minimization.

One immediate consequence of the Z-minimization is that, since (6.5.12) is a rational function of the parameters, the minimum value is an algebraic number times π^3. Thus, the ration of the volume of a toric Sasaki-Einstein manifold and that of S^5 is never a transcendental number. One can plug the toric data for $Y^{p,q}$ spaces, (6.4.9), into the formula above and check that it reproduces the volume given in (6.3.22).

As an example, let us carry out the minimization above for the $Y^{p,q}$ spaces. Let us parametrize $R = u + xv + yw$, then we have

$$Z(R) = \frac{p\left(p^2 - p\,q\,(1+x) + q\,(q\,x+2\,y)\right)}{(p+p\,x-y)\,y\left(p^2\,x - p\,(1+x)\,(q\,x+y) + (q\,x+y)^2\right)} \tag{6.5.18}$$

Its maximum is at

$$x = 0, \qquad y = \frac{p}{2q}\left(-2p + 3q + \sqrt{4p^2 - 3q^2}\right), \tag{6.5.19}$$

which reproduces the Reeb vector (6.4.12) correctly. One can check that it also reproduces the volume (6.3.22).

Chapter 7

Corresponding Quivers

7.1 Introductory remark

As seen in the previous chapters, the gauge theory on the D3-branes at the tip of a Calabi-Yau cone should have the cone itself as a part of its moduli space of vacua. The Ricci-flatness of the cone is difficult to analyze, since it involves the determination of the Kähler form which is not protected by supersymmetry. The cone as the complex manifold, on the contrary, is easier to analyze because of the relation

$$\frac{\{F = 0, D = 0\}}{G} \simeq \frac{\{F = 0\}}{G_{\mathbb{C}}}, \tag{7.1.1}$$

where $F = 0$ and $D = 0$ denote the F-flatness and the D-flatness conditions symbolically, G and $G_{\mathbb{C}}$ the gauge group and its complexification. The left hand side is the moduli space of the supersymmetric theory as determined directly by the Lagrangian, and the equality is as complex manifolds. Thus the complex structure of the moduli is determined by the superpotential and the gauge action alone, both of which receives no perturbative correction.

Toric cones have been the main focus of the analysis so far, since the existence of $U(1)^3$ action greatly facilitates it. Indeed, any toric cone can be obtained from the orbifold of $\mathbb{C}^3$ by a sequence of partial resolutions. The gauge theory probing the orbifold can be determined by an algorithm which is now standard [21], while the partial resolution corresponds to the Higgsing of the fields. Thus, as exemplified in [69], it is possible to obtain the gauge theory for any toric cone algorithmically, but it is extremely unwieldy and tedious even for a toric cones with only a few edges. It is now supplanted by the so-called brane tiling method which is convenient for the concrete calculation (see e. g. [70]). The method was motivated from the brane box models and was, for a while, a working rule of thumb rather than a logical development. Fortunately we now have a number of works which fills the logical gap and which makes the method to be based on sound physical [71] and mathematical [72] footings.

Once one finds the gauge theory which has the given cone as the moduli, the next task will be to study whether other properties match, for example the central charge a and the inverse of the volume. In this aspect an important property was found in [17], in which it was shown that for the pair of the toric data and the corresponding quiver constructed as above, the result of a-maximization and of Z-minimization always agree.

The aim of this chapter is to review some of these fascinating developments. We will first explicitly analyze the $Y^{p,q}$ spaces in section 7.2, and secondly explain the generalization to arbitrary toric cones 7.3 without presenting the reasoning behind the rules. We hope the section 7.2 gives sufficient credence in the reader to accept the results in 7.3 for the sake of this thesis; the detail can be found in

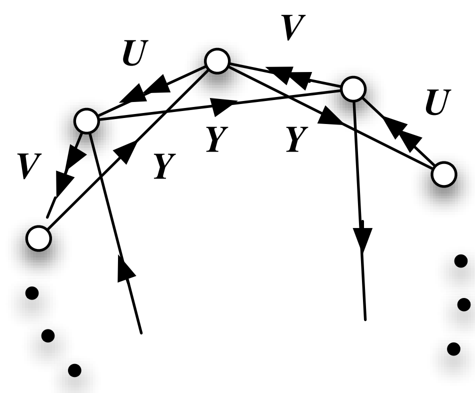

Figure 7.1: The quiver diagram for $Y^{p,p}$.

the references [71, 72].

7.2 Examples: $Y^{p,q}$

As the examples we use the cone over $Y^{p,q}$. It has $U(1)^2 \times SU(2)$ as the isometry, which has $U(1)^3$ as the subgroup. Thus it is a toric cone. The generators a, b, c and d of the toric cone are given by

$$a = u, \qquad b = u - v, \qquad c = u + pw, \qquad d = u + v + (p - q)w \tag{7.2.1}$$

where u, v and w are the generators of $U(1)^3$ symmetry, as described in (6.4.10). They are depicted in fig. 6.4. Note that we renamed the label of the vertices.

7.2.1 Construction

As is mentioned in section 6.4, $Y^{p,p}$ is a $\mathbb{Z}_{2p}$ orbifold of S^5, while $Y^{p,0}$ is a $\mathbb{Z}_p$ orbifold of $T^{1,1}$. Indeed, let (z_1, z_2, z_3) parametrize a flat $\mathbb{C}^3$, and divide by the action of $\mathbb{Z}_{2p}$ which sends

$$(z_1, z_2, z_3) \mapsto (\alpha^2 z_1, \alpha^{-1} z_2, \alpha^{-1} z_3) \tag{7.2.2}$$

where α is a $2p$-th root of unity. One can easily check that the corresponding orbifold $\mathbb{C}^3/\mathbb{Z}_{2p}$ has the toric diagram of $Y^{p,p}$. For $Y^{p,0}$, we need to divide the conifold

$$xy = zw \tag{7.2.3}$$

by the action of $\mathbb{Z}_p$ given by

$$(x, y, z, w) \mapsto (\beta x, \beta^{-1} y, \beta z, \beta^{-1} w). \tag{7.2.4}$$

Here β is a p-th root of unity.

Given the action of the orbifold group $\mathbb{Z}_{2p}$ (7.2.2), one can determine the structure of the quiver theory on the D3-branes which probe the tip of the cone. It has $2p$ nodes $SU(N)_i$ ($i = 1, \ldots, 2p$), and there are several kinds of bifundamental fields. U^α connects $SU(N)_i$ to $SU(N)_{i+1}$ for even i, V^α connects $SU(N)_i$ to $SU(N)_{i+1}$ for odd i, and Y connects $SU(N)_i$ to $SU(N)_{i-2}$. Here α is the index for the doublets of the $SU(2)$ global symmetry, where the addition and subtraction in the index i are taken to be of modulo $2p$. The quiver diagram is depicted in figure 7.1.

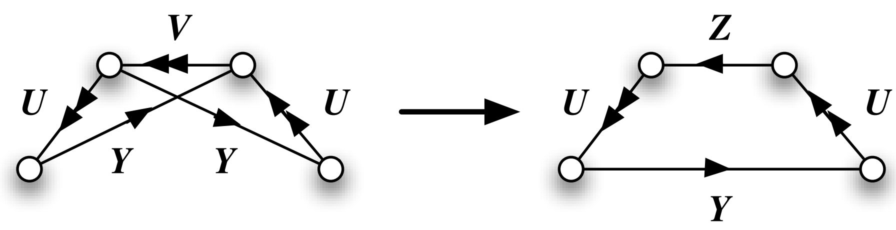

Figure 7.2: Change to make the $Y^{p,q-1}$ quiver from the $Y^{p,q}$ quiver.

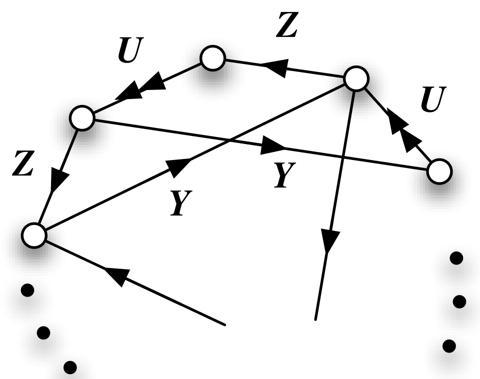

Figure 7.3: The quiver diagram for $Y^{p,0}$.

In [14] the authors proposed a method to construct the quivers for $Y^{p,q}$. The proposal for obtaining the quiver for $Y^{p,q-1}$ from $Y^{p,q}$ is to change the local structure of the quiver as exemplified in the figure 7.2, that is, to take consecutive four nodes $SU(N)_{i,i+1,i+2,i+3}$ connected by the fields U^α, V^α and again by U^α. One drops the field V^α connecting the node $i+1$ to the node $i+2$ and the Y fields connecting $i+3$ to $i+1$ and $i+2$ to i. One then adds the field Z connecting $i+1$ to $i+2$ and another field Y connecting $i+3$ to i.

There are several places on the quiver diagram where the modification mentioned above can be done. Thus the quiver corresponding to the $Y^{p,q}$ toric cone is not uniquely determined. It is known, however, that the differing quivers are related to each other by the chain of Seiberg dualities. We can construct quiver diagrams for $Y^{p,q}$ starting from the known $Y^{p,p}$ quiver and reducing q one by one. After one replaces every occurrences of edges of V^α by the prescription, one gets the quiver diagram depicted in figure 7.3. One can check that the moduli space of that quiver is precisely the conifold divided by the action (7.2.4), that is, the cone over $Y^{p,p}$.

The superpotential is given by

$$W = \sum (-1)^s \operatorname{tr} U^\alpha V_\alpha Y - \sum \operatorname{tr} U^\alpha Y U_\alpha Z \tag{7.2.5}$$

where the first sum is over the triangles connecting i, $i+1$ and $i+2$, and the sign $(-)^s$ is chosen to be the same as that of $(-)^i$. The second sum is over the quadrangles introduced by the modification above.

	#	a	b	c	d	B
U_1	p	0	1	0	0	$-p$
U_2	p	0	0	0	1	$-p$
V_1	q	1	1	0	0	q
V_2	q	1	0	0	1	q
Y	$p+q$	0	0	1	0	$p-q$
Z	$p-q$	1	0	0	0	$p+q$
θ	—	1/2	1/2	1/2	1/2	—

Table 7.1: Multiplicities and global symmetry charges for $Y^{p,q}$ quiver.

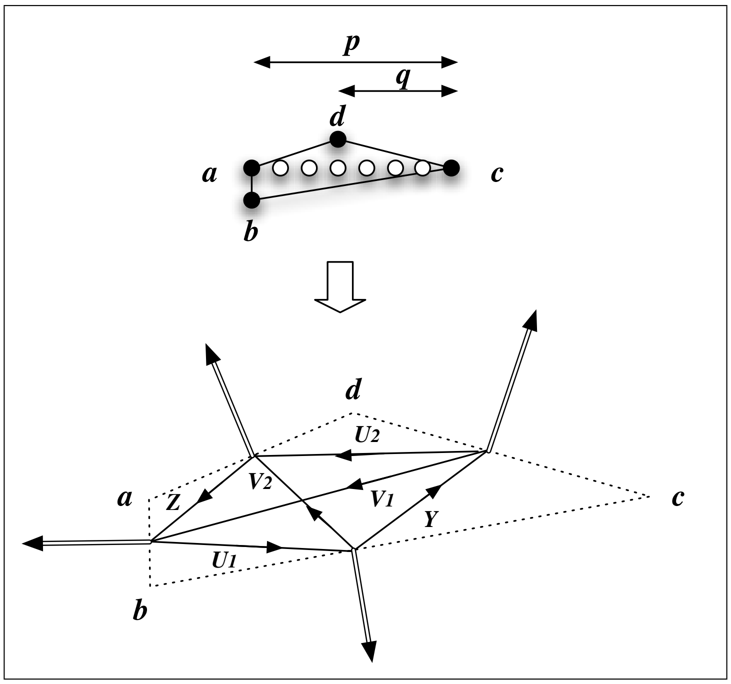

Figure 7.4: The 'folded-quiver' description of the $Y^{p,q}$ quiver.

7.2.2 Moduli

In the original paper [14] cited above, not much reasoning was given for the rule of the modification of the quivers. Rather, the matching of the central charge a calculated from the a-maximization and the inverse of the volume was used as the justification. The matching of the moduli space to the cone was done afterwards in a series of papers, culminating in the general prescription given in [70]. Here we would like to take a slightly more logical approach, by first checking the matching of the moduli space and the cone. The presentation basically follows that in [73], although the motivation in it was slightly different from ours.

We begin by studying the global symmetry of the quiver theory constructed in the previous subsection. By some calculation, one finds that the maximal number of the commuting $U(1)$ charges are four. We call them $U(1)_{a,b,c}$ and $U(1)_d$. The charges of the bifundamental fields and the supercoordinates θ_α are tabulated in table 7.1, together with their multiplicities. One can check by a direct calculation that $U(1)_{a,b,c,d}$ are indeed conserved charges.

The basis of the charges, consisting of a, b, c and d, is distinguished in the sense that they arise from the 'folded quiver' description, see figure 7.4. There, one draws edges connecting the midpoint of the edges of the toric data. The edges thus drawn inside the toric diagram are assigned to the bifundamental fields as in the figure. The vertices of the original toric diagram correspond to the global $U(1)$ symmetries of the quiver, and the charge is assigned so that a field corresponding to the edge E has charge 1 under the symmetry corresponding to the vertex V if V lies on the right hand side of E, is neutral otherwise. The supercoordinates θ_α have charge one half for every symmetry corresponding to a vertex. Now, the vertices a, b, c and d satisfy the relation

$$0 = (p+q)a - pb + (p-q)c - pd \tag{7.2.6}$$

as the generators of the toric cone. The corresponding $U(1)$ symmetry B, defined by the right hand side of the above equation, that is,

$$B = (p+q)a - pb + (p-q)c - pd \tag{7.2.7}$$

is called the baryon symmetry. It is also tabulated in table 7.1.

Let us move on to the study of the moduli space itself. The gauge theory for one D3-brane is represented by the quiver diagram where every node is taken to represent a $U(1)$ gauge group. The moduli space $\mathcal{M}$ should then correspond to the transverse movement of the D3-brane. For N D3-branes, the nodes are taken to be $SU(N)$, and the moduli space for them automatically contains $\mathcal{M}^N/\mathfrak{S}_N$ as a subspace, although other branches may exist. What needs to be shown is that $\mathcal{M}$ is the cone over $Y^{p,q}$ as a complex manifold.

A complex manifold is defined by gluing patches, and they in turn is determined by the totality of the holomorphic functions (without poles) on them. A holomorphic function corresponds to a gauge invariant operator of the theory considered, modulo the F-flatness conditions. Without the Fayet-Iliopoulos term as in the present case, the moduli space is automatically conic and covered by one patch.

Gauge invariant operators for quiver gauge theories with $U(1)$ as nodes are closed loops on the quiver, since the gauge $U(1)$ charges for the bifundamental fields should cancel out. The global $U(1)$ charges do not necessarily cancel. One can check that the baryon charge B is always zero for closed loops, while there are three operators with linearly independent charges under a, b, c and d. Thus the moduli space is acted by three $U(1)$ symmetries which span the hyperplane $B = 0$. The dimension of the moduli space itself can be determined by examining the number of the fields and the number of the F- and D-flatness conditions. One finds it is three-dimensional. Thus $\mathcal{M}$ has as many $U(1)$ symmetry as its dimension, which is the definition of the toric manifold. As was discussed $\mathcal{M}$ is conic and covered by one patch. Such toric manifold is called affine, and it is determined by the charges of the available holomorphic functions. Indeed, one can check that two gauge invariant operators with the same $U(1)$ charges are always equal modulo the F-term conditions. The charges of the gauge invariant operators form a cone in a three-dimensional lattice, and its dual cone is the cone specified by the toric data.

More specifically, we know from the table 7.1 that the charges a, b, c and d of any composite operators constructed from chiral fields are positive, which form a cone with four edges in $\mathbb{Z}^4$. Gauge invariant operators are inside the cone and fall on the hyperplane $B = 0$. Thus they form a cone with four edges in $\mathbb{Z}^3$, which is the lattice points on the hyperplane $B = 0$.

The toric data for the cone over $Y^{p,q}$ were given in (7.2.1). Its dual cone, which should be the cone of the available charges, can be realized in the hypersurface $B = 0$. The dual cone is defined by the condition

$$a, b, c, d \geq 0. \tag{7.2.8}$$

Thus, the generators of the dual cone are found by looking for lattice points on $B = 0$ with two entries zero. They are

$$(a, b, c, d) = (0, 0, p, p-q), \quad (p, 0, 0, p+q), \quad (p, p+q, 0, 0), \quad \text{and} \quad (0, p-q, p, 0). \tag{7.2.9}$$

Gauge invariant operators with the charges specified as above can be found easily. Namely, if we follow the quiver diagram counterclockwise on the outermost edges, we have a loop consisting of p U's, q Z's and $(p-q)$ V's, in which operators with charge $(p, 0, 0, p+q)$ and $(p, p+q, 0, 0)$ can be found. If we follow the diagram clockwise using the inner edges Y, the loop can be closed by using p Y's and $(p-q)$ U's, in which the charge combinations $(0, 0, p, p-q)$ and $(0, p-q, p, 0)$ are realized. If we would like to show that $\mathcal{M}$ is the cone over $Y^{p,q}$ in full rigor, we have to show that any lattice points inside the cone defined by (7.2.8) are realized as the charge combination of the gauge invariant operators; we would like to content ourselves by having the realization of the generators of the cone as above.

7.2.3 Central charge

We would like to determine the central charges for the $Y^{p,q}$ quiver theory. Let us redefine the basis of the global $U(1)$ charges to that formed by

$$F = a - c, \quad R_0 = 2c, \quad \sigma_3 = b - d \tag{7.2.10}$$

and the baryon symmetry B. σ_3 is the Cartan generator of the $SU(2)$ global symmetry under which U^α and V^α are doublets, thus it does not enter the superconformal R-symmetry.

Let us now introduce the trial R-symmetry $R = R_0 + s_F F + s_B B$. The trial a function is

$$\begin{aligned}\frac{9}{32}\operatorname{tr} R^3 = \frac{9N^2}{32} \times \Big(&2p + 2p(-1 - ps_B)^3 + 2q(-1 + s_F + qs_B)^3 + \\ &(p+q)(1 - s_F + (p-q)s_B)^3 + (p-q)(-1 + s_F + (p+q)s_B)^3\Big).\end{aligned} \tag{7.2.11}$$

Its maximum is at

$$s_F = \frac{2p + 3q - \sqrt{4p^2 - 3q^2}}{3q}, \qquad s_B = \frac{2}{3q^2}\left(\sqrt{4p^2 - 3q^2} - 2p\right), \tag{7.2.12}$$

and the value of the a-function at the maximum is

$$a_{p,q} = N^2 \frac{p^2}{4q^4}\left(\sqrt{4p^2 - 3q^2}^{\,3} - 8p^3 + 9pq^2\right). \tag{7.2.13}$$

The volume of $Y^{p,q}$ was presented in (6.3.22), and it satisfies the relation

$$a_{p,q} = \frac{N^2}{4}\frac{\pi^3}{\operatorname{Vol} Y^{p,q}}, \tag{7.2.14}$$

as it should be from the AdS/CFT correspondence, (6.2.10). This is a satisfying test of the Maldacena conjecture. Indeed, the calculation above for $a_{p,q}$ is just the maximization of a cubic polynomial, motivated by the field theory consideration, while the construction and the calculation of the volume of $Y^{p,q}$ are a difficult problem in classical general relativity. Two calculations have no direct relationship whatsoever, without the light of string theory.

7.3 Generic properties

In this section we abstract the properties we saw in the previous section to the gauge theories on the D3-branes which probe generic toric Calabi-Yau cones. The structure of the quiver theory consists of three parts, namely the gauge group, the matter representations and finally the superpotential. In the toric case a technique is now available [74, 75], which enables us to accomplish the most difficult part, which is the determination of the superpotential of the quiver theory. Fortunately, it is not necessary to know the superpotential for the purpose of this thesis. We only need the gauge groups and the bifundamentals connecting them, which we summarize below.

Let us consider a toric Calabi-Yau cone with the toric data $k_I = (1, \vec{k}_I)$ with $I = 1, 2, \ldots, d$. We can guess the following properties from the discussion in the previous section on $Y^{p,q}$:

1. The gauge group is $SU(N)^{\mathcal{A}}$, where $\mathcal{A}$ is twice the area of the polygon whose vertices are the toric data k_I.

2. The bifundamental chiral superfields can be grouped in $d(d-1)/2$ sets, which we can call $\mathcal{B}_{ij}$, where i and j label two midpoints of the edges of the toric diagram.

3. In each set $\mathcal{B}_{ij}$ there are
$$|\mathcal{B}_{ij}| = \det(\vec{v}_1, \vec{v}_2) \tag{7.3.1}$$
of bifundamental fields, where $\vec{v}_i = \vec{k}_{i+1} - \vec{k}_i$.

4. Global symmetries Q_I are in one-to-one correspondence with the generators k_I of the toric cone.

5. The fields in $\mathcal{B}_{ij}$ is charged with charge one or zero under the global symmetry Q_I if the vertex k_I is on the right or left hand side of the arrow connecting i to j, respectively.

6. The supercoordinate is charge $1/2$ under Q_I for any I.

7. Of d $U(1)$ symmetries, only three linear combinations, which correspond to the three $U(1)$ rotations of the toric cone, act non-trivially on the gauge invariant operators formed from the closed loops of the quiver. The surviving combination is given by the map
$$a^I Q_I \mapsto a^I k_I. \tag{7.3.2}$$

Before proceeding let us comment on what is known about the validity of the various properties: Property 1 is a well established fact. The total number of gauge groups is equal to the total number of compact cycles (0-, 2- and 4-cycles) in the completely resolved Calabi-Yau cone. Since there is no odd-homology, this number is the Euler number of the resolved non-compact Calabi-Yau, which is, in turn, known to be given by twice the area of the toric diagram. Properties 2 and 3 were proposed in [76], under the name of "folded quiver", and was inspired from a T-dual description of the Calabi-Yau space and the branes probing it. Namely, the dual diagram of the toric diagram represents the (p, q) five-brane web, and the edges $\vec{v}_i$, when rotated by 90 degrees, represents the direction of the infinite (p, q) 5-brane. Two such 5-branes will intersect at $\det(\vec{v}_i, \vec{v}_j)$ points, yielding the same number of bifundamental matter fields. Each brane will also host a $U(1)$ symmetry, which is seen as the global symmetry from the point of view of the D3-branes on the original Calabi-Yau cone. These are Properties 4 and 5.

Property 3 does not hold for all toric phases of the quiver. We expect there is always at least one toric phase where the number of the fields is precisely given by the determinant (7.3.1). This is known to be the case for the set of theories $Y^{p,q}$ and $L^{p,q|r}$. For the $Y^{p,q}$'s all toric phase have been classified [77], and in some phases, with so-called double impurities, property 3 does not hold as stated. In these cases there are additional pairs of fields with opposite charges. It was also shown for toric del Pezzo surfaces in [78]. We expect it to be possible to give a general proof studying intersection numbers of compact three-cycles in the mirror Calabi-Yau, as was conjectured in [76] on the base of [78]. For recent work see [79, 71, 80].

The strong evidence for the validity of Properties 5, 6 and 7 listed above was given in the work of Butti and Zaffaroni [17, 80], where it was shown that the field theory computation of the cubic 't Hooft anomaly c_{RRR} matches precisely with the geometric results for the volumes of the Sasaki-Einstein manifolds, as expected from the AdS/CFT correspondence. The volumes on the gravity side can be computed using the results of Martelli, Sparks and Yau [15], which enables us to compute the volumes just in terms of toric data. One of the main results in this thesis, presented in the next chapter, is that the cubic 't Hooft anomalies c_{IJK} calculated using the properties 5, 6 and 7 match with the Chern-Simons coefficients as computed from gravity.

As an aside, let us note that, in addition to 't Hooft anomalies one can readily compute the scaling dimension of dibaryon operators, using the "folded quiver" picture. This gives additional evidence for the validity of properties $1, 2$ and 3. Also the topology of some supersymmetric three-cycle can be matched with this picture [70].

Properties 5 and 6 imply that the gauginos have charge one half and their contribution to cubic anomalies is always $\mathcal{A}N^2/8$. The fermionic component of the bifundamental superfields have charge $-1/2$ or $1/2$. We thus see that in this way all the charges are half integral, and every bifundamental field contributes $\pm N^2/8$ to the cubic anomalies. The point is that this basis is easy to be identified in the gravity dual. Indeed, the dibaryon constructed from the field in $\mathcal{B}_{I,I+1}$ has the charge $\delta_{IJ}N$ under the symmetry Q_J, which will be matched against the charge of the D3-brane which wraps the invariant three-cycle in the Sasaki-Einstein manifold.

Chapter 8

Triangle Anomalies from Einstein Manifolds

Most of the results in this chapter first appeared in the work [19] by S. Benvenuti, L. A. Pando Zayas and the author of the thesis, except the result in section 8.6 which is based on the papers [17, 81].

8.1 Introductory remark

Since this chapter is rather long, we would like to start with the introduction to the chapter. As we saw in section 4.5, the global symmetry on the CFT side corresponds to the gauge fields on the AdS side, and the triangle anomalies among the global symmetries in CFT translates to the Chern-Simons (CS) couplings $(24\pi^2)^{-1}\int c_{IJK}A^I \wedge F^J \wedge F^K$ for the five-dimensional gauge fields, and the matching between them provides a quantitative check of the AdS/CFT correspondence. It was carried out in [3] for $X = S^5$ using supergravity results of [82, 83], but it has not yet been done for other Einstein manifolds. It is well-known that triangle anomalies can be extracted by a simple one-loop computation in the gauge theories, and that they are topological objects. We thus expect that it should be possible to develop a generic quantitative understanding also on the gravity side of the duality, because they should belong to "protected sectors" of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. Accomplishing it is the main objective of this chapter.

Other types of "protected sectors" of the AdS/CFT correspondence are given by the Bogomol'ny-Prasad-Sommerfield (BPS) operators, which are protected by supersymmetry. In this case one can map the scaling dimensions of the BPS operators to the energy of the corresponding BPS states in the type IIB string theory on $\mathrm{AdS}_5 \times X$. We can expect it to be possible to understand the dual BPS objects on the gravity side in general, without the need of having the explicit metrics. This is indeed the case, for instance, for dimensions of baryonic BPS operators, corresponding to the volumes of supersymmetric (SUSY) cycles, which can be computed with the procedure uncovered in [15]. In the same way, we expect that the CS coefficients can be calculated on the gravity side without the knowledge of the explicit metrics.

The 5d Chern-Simons coefficients also appear prominently in the analysis of M-theory on Calabi-Yau threefolds. There, the supergravity reduction results in the formula

$$c_{IJK} \propto \int_{\mathrm{CY}} \omega_I \wedge \omega_J \wedge \omega_K, \tag{8.1.1}$$

where $\omega_{I,J,K}$ are harmonic two-forms on the Calabi-Yau. At this stage, the formula for c_{IJK} is difficult to evaluate, since finding harmonic forms is hard. One can show, however, that c_{IJK} does not change when one shifts ω_I by an exact form. Thus, the determination of ω_I as a cohomology class suffices in the calculation of c_{IJK}. Indeed, they are given in terms of the triple intersections of three four-cycles of the Calabi-Yau, and are integers. c_{IJK} should be integers for reasons also from the five-dimensional point of view, which is the requirement of the five-dimensional action to be defined modulo 2π on a spacetime with nontrivial topology. We expect to find a similarly robust formula for the CS coefficients in the case of the type IIB supergravity on compact, positively curved, Einstein manifolds X.

Thus, our first objective is to obtain a geometrical formula for the Chern-Simons coefficients c_{IJK} for the type IIB supergravity on $\mathrm{AdS}_5 \times X$. The result we will obtain is so elegant that we would like to give the formula here. It is given by

$$c_{IJK} = \frac{N^2}{2} \int_X \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}}\,, \tag{8.1.2}$$

where three-forms ω_I of X are the internal wavefunctions for the gauge fields on AdS_5, and the Killing vectors k_I is determined by ω_I. N is the number of the self-dual five-form flux through X. We will show that this formula gives robust topological quantities in a precise sense. In particular, explicit knowledge of the Einstein metric on X is not necessary to evaluate the formula (8.1.2).

We provide an explicit evaluation of c_{IJK}, through (8.1.2), for large sets of Sasaki-Einstein manifolds, namely, circle bundles over del Pezzo surfaces and toric Sasaki-Einstein manifolds. We will find complete agreement on the gravity side and the field theory side. For toric SE we obtain

$$c_{IJK} = \frac{N^2}{2} |\det(k_I, k_J, k_K)| \tag{8.1.3}$$

where $k_I \in \mathbb{Z}^3$ is the I-th toric datum. In other words, c_{IJK} is simply given by the area of a triangle formed by the three toric data. We recover the formula (8.1.3) from field theory, thus providing a very general check of AdS/CFT.

With the formula for c_{IJK}, one can form the trial a-function to find the central charge, while with k_I one can use the Z-minimization to find the volume of the base Sasaki-Einstein manifold. We will see how these two extremization are related, utilizing the formula (8.1.3).

We will also analyze the BPS operators which are related to giant gravitons, emphasizing the interplay between objects protected by supersymmetry and topological properties of X. Throughout the analysis, we will see that there is an intricate mixing of the angular momenta and baryonic charges, which reflects the fact that the D3-branes wrapping three-cycles in the SE manifold is partly a giant graviton.

The organization of this chapter is the following: first we describe in section 8.2 the supergravity reduction which gives the formula for the CS terms and gauge coupling constants. Then, we discuss the normalization of gauge fields and the charges in section 8.3, where we will see that the formula for the CS terms is topological in a precise sense. We evaluate the formulae for toric Sasaki-Einstein manifolds and for the circle bundles over del Pezzo surfaces in section 8.4. In section 8.5, we turn to field theory dual and show, firstly based on explicit examples, that the results obtained in previous sections match with predictions based on $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. We also provide a generic proof for toric cases. We present in section 8.6 the beautiful analysis by [17, 81] which compares the a-maximization and the Z-minimization, utilizing the formula obtained in section 8.5. In section 8.7, we explain the simplicity of our results in section 8.5 using the flow triggered by the Higgsing of dibaryons. Finally in section 8.8, we elaborate on the mathematics behind the charge lattice associated to the five-dimensional Einstein manifold with isometries.

8.2 Perturbative Supergravity Reduction

Consider the type IIB theory on $\mathrm{AdS}_5 \times X$ where X is an Einstein manifold of dimension five. Let us carry out the Kaluza-Klein reduction and retain only the massless gauge fields. The corresponding five-dimensional action has the form

$$S = \frac{1}{2}\int \tau_{IJ} F^I \wedge *F^J + \frac{1}{24\pi^2}\int c_{IJK} A^I \wedge F^J \wedge F^K + \cdots , \tag{8.2.1}$$

which yields the equation of motion

$$\tau_{IJ} d * F^I = \frac{1}{8\pi^2} c_{IJK} F^J \wedge F^K. \tag{8.2.2}$$

We would like to calculate the Chern-Simons coefficient c_{IJK} of the gauge fields. We will eventually choose the indices $I, J, \ldots$ to label the integral basis of the gauge fields in the next section, but in this section we take them arbitrarily. We chose the numerical coefficient $(24\pi^2)^{-1}$ so that $c_{IJK} = \operatorname{tr} Q_I Q_J Q_K$ under the AdS/CFT correspondence, where Q_I is the global symmetry corresponding to the gauge field A_I, and the trace is over the label of Weyl fermions.

The arguments which are to be presented in sections 8.2.1, 8.2.2 and 8.2.3 only uses the fact that the metric is Einstein, so it is applicable, e.g. to the manifolds $T^{a,b}$ for $(a, b) \neq (1, 1)$.

8.2.1 Details of supergravity reduction

Our goal in this section is to perform a compactification of a ten-dimensional solution of IIB supergravity to five dimensions. This subsection is highly technical, so that if a reader wants only the outcome s/he can skip this section and go directly to section 8.2.2.

There is an extensive literature on supergravity reduction on positively curved symmetric manifolds. For example, there are some constructions of full consistent non-linear Ansatz for the reduction on the spheres [84]. Other interesting truncations are presented in [85] and references therein. In this subsection we carry out the compactification of the type IIB theory on generic 5-dimensional Einstein manifolds. It is worth stressing that we are not attempting a consistent truncation to a five-dimensional theory. As such, we are forced to the perturbative analysis and will not pursue full non-linear reduction in this paper. Indeed, it is known that the consistent truncation is possible only for a restricted set of manifolds [86].

Consider the type IIB theory compactified on an Einstein 5-manifold X to have five-dimensional theory on AdS_5. Let the coordinates of X and AdS be y^i and x^μ, and their fünfbeine be e^i and f^μ, respectively.

Since the action of the self-dual five-form in ten dimensions is rather subtle, we carry out the Kaluza-Klein analysis at the level of equation of motion[1]. Let us explain the main technical point before going into the details. Schematically, one first expands the fluctuation using the harmonics of the internal manifold X,

$$\phi(x, y) = \phi_0(x, y) + \delta\phi^{(i)}(x)\psi_{(i)}(y) + \cdots , \tag{8.2.3}$$

so that $\delta\phi^{(i)}$ are the mass eigenstates. Then, one can identify the cubic couplings such as the CS coefficient by finding the equation of motion of $\delta\phi^{(i)}$ in the form

$$(D^2 - m^2)\delta\phi^{(i)} = C^{(i)}_{(j)(k)}\delta\phi^{(i)}\delta\phi^{(j)} + \cdots . \tag{8.2.4}$$

[1] Quite recently, the authors of [87] constructed a covariant action for the self-dual five-form fields, and can be used to derive the results presented in this section transparently. Unfortunately, their form of the action needs a choice of the Lagrangian subspace of the space of the five-forms, and it takes some pages to present their formalism. Thus, we chose to express the results via a more down-to-earth approach taken in [19].

If one is only interested in obtaining certain parts of the cubic coupling, one can set to zero any fluctuation which does not multiply the couplings. It does not change the results, and at the same time it greatly reduces the calculational burden.

Another technical difficulty lies in maintaining the self-duality of the Ansatz for the five-form. Suppose X has ℓ $U(1)$ isometries k^i_a, $a = 1, \ldots, \ell$ so that $\exp(2\pi k^i_a \partial_i)$ is the identity. For toric SE manifolds, $\ell = 3$. The ansatz for the metric is the usual one,

$$ds^2_X = \sum_i (e^i + k^i_a A^a)^2, \tag{8.2.5}$$

where e^i are the fünfbein forms of the Einstein manifold and $A^a = A^a_\mu dx^\mu$ are one-forms on AdS$_5$.

Let us abbreviate $\hat{e}^i = e^i + k^i_a A^a$. Then, the Hodge star exchanges

$$f^1, \ldots, f^5 \quad \longleftrightarrow \quad \hat{e}^1, \ldots, \hat{e}^5. \tag{8.2.6}$$

Thus, one can anticipate that the introduction of the following $\hat{}$ operation on general differential forms of X defined by replacing e by $\hat{e}$,

$$\alpha^{(p)} = \alpha_{i_1 \cdots i_p} e^{i_1} \cdots e^{i_p} \mapsto \hat{\alpha}^{(p)} \equiv \alpha_{i_1 \cdots i_p} \hat{e}^{i_1} \cdots \hat{e}^{i_p}, \tag{8.2.7}$$

greatly helps in maintaining the self-duality of the Ansatz for F_5.

The following two formulae are useful in calculation. First is a formula for the $\hat{}$ operation using interior products:

$$\hat{\alpha} = \alpha + A^a \wedge \iota_{k_a} \alpha + \frac{1}{2} A^a \wedge A^b \wedge \iota_{k_b} \iota_{k_a} \alpha + \cdots . \tag{8.2.8}$$

Another is $*(\alpha^{(5-p)} \wedge \beta^{(p)}) = (-)^p (*\alpha) \wedge *\beta$ where the number in the parentheses in the superscript denotes the degree of the forms.

Let us carry out what we have just outlined. The equations of motion and the Bianchi identity in the type IIB supergravity are:

$$R_{\mu\nu} = \frac{c}{24} F_{\mu\alpha\beta\gamma\delta} F_\nu{}^{\alpha\beta\gamma\delta}, \tag{8.2.9}$$

$$F = *F, \tag{8.2.10}$$

$$dF = 0 \tag{8.2.11}$$

where $R_{\mu\nu}$ is the Ricci curvature of the ten-dimensional metric and

$$F = \frac{1}{5!} F_{\mu\nu\rho\sigma\tau} dx^\mu dx^\nu dx^\rho dx^\sigma dx^\tau \tag{8.2.12}$$

is the self-dual five-form field strength. We use the relation above with $p!$ substituted for $5!$ to translate a p-form to its components in general. c is equal to $16\pi^6 \alpha'^4 g_s^2$ as before. We set other form fields and fermions to zero, and dilaton to constant.

Let N units of five-form flux penetrate X, where we normalize the five-form F_5 to have $\int F_5 \in 2\pi\mathbb{Z}$. Then, the zero-th order solution is

$$ds^2 = L^2 ds^2_{\text{AdS}} + L^2 ds^2_X, \qquad F = \frac{2\pi N}{V} (\text{vol}_X + \text{vol}_{\text{AdS}}) . \tag{8.2.13}$$

We take the convention $R_{\mu\nu} = -4g_{\mu\nu}$ for the AdS part, $R_{mn} = 4g_{mn}$ for the SE part. V is the volume, $V = \text{Vol}(X)$. Plugging (8.2.13) in to the equation of motion of the metric, we get

$$4 = c \left(\frac{2\pi N}{V} \right)^2 L^{-8}. \tag{8.2.14}$$

Let us expand the fluctuation around the zero-th order solution in modes. One can consistently set to zero all the modes which are not invariant under the $U(1)$ isometries. We then take the ansatz for F_5 as

$$\frac{V}{2\pi N}F_5 = \hat{e}^1 \cdots \hat{e}^5 + B^a \wedge *k_a - F^I \wedge \hat{\omega}_I + *F^I \wedge \widehat{*\omega_I} + (*B^a) \wedge k_a + f^1 \cdots f^5 \tag{8.2.15}$$

where $k_a = g_{ij}k^i_a dy^j$, ω_I are three-forms to be identified shortly, $B^a = B^a_\mu dx^\mu$ and $F^I = F^I_{\mu\nu}dx^\mu dx^\nu/2$. We will see that this gives consistent equation of motion in five dimensions.

F_5 above satisfies $F_5 = *F_5$ by construction, because the one-forms f^μ and $\hat{e}^i$ constitutes the zehnbein of the metric. $dF_5 = 0$ requires

$$d\omega_I = c^a_I \iota_{k_a} \operatorname{vol}_X \tag{8.2.16}$$

for some constants c^a_I. We define $k_I \equiv c^a_I k_a$ for brevity. Furthermore, we assume ω_I to be co-closed. Then, $dF_5 = 0$ imposes on B^a, F^I the equations

$$d(A^a + B^a) = c^a_I F^I, \tag{8.2.17}$$

$$dF^I = 0, \tag{8.2.18}$$

$$d(*F^I) \wedge *\omega_I = -(*B^a) \wedge dk_a + F^I \wedge F^J \wedge \iota_{k_J}\omega_I \tag{8.2.19}$$

where we kept the fluctuations up to the second order. Let us define ω_a by $*dk_a/8$. One has $d\omega_a = \iota_{k^a}\operatorname{vol}$ by using the fact[2] that we have $*d*dk = 2tk$ for any Killing vector k in an Einstein spaces with $R_{ij} = tg_{ij}$. Then we see, from (8.2.19),

$$d*F^I \wedge \omega_K \wedge *\omega_I = -8*B^a \wedge \omega_K \wedge *\omega_a + F^I \wedge F^J \wedge \omega_K \wedge \iota_{k_J}\omega_I. \tag{8.2.20}$$

Another important equation of motion comes from the Ricci curvature $R_{\mu\hat{i}}f^\mu\hat{e}^i$ with one leg in the AdS and one leg in the SE. While

$$R_{\mu\hat{i}} = \frac{1}{2}k_{ia}\nabla_\nu(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \tag{8.2.21}$$

from (8.2.5), the right hand side of (8.2.9) is given by

$$\frac{c}{24}F_{\mu\cdots\cdot}F_{\hat{i}}{}^{\cdots\cdot} = \frac{c}{24}\left(\frac{2\pi N}{V}\right)^2 L^{-8}\left(48B^a_\mu k_{ai} - 6(*F^I)_{\mu\nu\rho}(*\omega_I)_{..}F^{J\nu\rho}(\omega_J)_{\hat{i}}{}^{..}\right) \tag{8.2.22}$$

$$= 8B^a_\mu k_{ai} - 4(*F^I \wedge F^J)_\mu \frac{(\omega_I \iota_{e_i}\omega_J)}{\operatorname{vol}}. \tag{8.2.23}$$

Thus we get

$$\frac{1}{16}(d*dA^a) \wedge \langle k_a k_K\rangle \operatorname{vol} = *B^a \wedge \langle k_a k_K\rangle \operatorname{vol} + \frac{1}{2}F^I \wedge F^J \wedge \omega_I \iota_{k_K}\omega_J \tag{8.2.24}$$

[2] One can replace ∂_i by ∇_i in the definition of Lie derivative. Thus $\nabla_i k_j + \nabla_j k_i = 0$. Then

$$R_{lj}k^l = R^k_{lkj}k^l = [\nabla_k, \nabla_j]k^k = g^{kl}[\nabla_k, \nabla_j]k_l = -g^{kl}\nabla_k\nabla_l k_j - g^{kl}\nabla_j\nabla_k k_l = -\nabla^2 k_j.$$

Hence, for Einstein manifold with $R_{ij} = tg_{ij}$, we have

$$(*d*dk)_i = \nabla^j(\nabla_i k_j - \nabla_j k_i) = 2tk_i.$$

where $\langle k_1, k_2\rangle$ of two vectors is their inner product.

From (8.2.20) and (8.2.24) we see that the mass eigenstate is B^a and the mixture of A^a and B^a, and the former is massive and the latter is massless. Let us add the both sides of the equations (8.2.20) and (8.2.24), and integrate over the internal manifold X. Using $\int_X \omega_K \wedge *\omega_a = \int_X \langle k_K k_a\rangle \operatorname{vol}/8$, the term including the massive mode B^a cancels, and we finally obtain the equation of motion for massless fields :

$$d * F^I \int_X (\omega_K \wedge *\omega_I + \frac{1}{16}\langle k_K k_I\rangle \operatorname{vol}) = \frac{1}{4} F^I \wedge F^J \int_X \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}} \tag{8.2.25}$$

where $\{IJK\} = IJK + IKJ + \cdots$ without $1/6$. The factor which multiplies $d * dF_I$ exactly reproduces the combination $g_{IJ}^{-2KK} + g_{IJ}^{-2CC}$ which appeared in ref [18], where it was derived in a slightly different way.

Let us recapitulate what happens during the detailed calculation above. If we reduce some higher-dimensional form-field theory on an internal manifold without isometries, we need to have simultaneously closed and co-closed wavefunctions in the internal manifold to have a massless field in the non-compact dimensions. If the metric is the sole dynamical field, then upon reduction an isometry produces a gauge field through the ansatz (8.2.29). Through the coupling between the metric and the five-form field, the gauge field from $g_{\mu\nu}$ and the gauge field from F_5 with co-closed but nonclosed wavefunctions get off-diagonal components in the mass matrix, and precisely one linear combination remains massless per one Killing vector field. Thus, the total number of massless gauge fields in AdS is

$$d = \ell + b^3, \tag{8.2.26}$$

where ℓ is the number of independent Killing vectors and b^3 is the dimension[3] of $H^3(X)$.

8.2.2 Summary of the reduction

The presentation in the previous subsection was precise but highly technical. We would like to summarize the result here in a more informal way. First we change the normalization of ω_I and k_I so that

$$(\omega_I \text{ from now on}) = -\frac{2\pi}{V}(\omega_I \text{ so far}), \tag{8.2.27}$$

and that

$$(k_I \text{ from now on}) = 2\pi(k_I \text{ so far}). \tag{8.2.28}$$

The definitions used in the previous subsection were chosen to minimize the number of messy factors of V and 2π in the equation of motion; the new ones are more suitable for the topological analysis later.

Suppose X has ℓ $U(1)$ isometries k_a^i, $(a = 1, \ldots, \ell)$. Take the usual Kaluza-Klein ansatz for the metric

$$ds_X^2 = \sum_i (e^i + k_a^i A^a)^2 \tag{8.2.29}$$

where e^i are the fünfbein forms of the SE, and A^a are the one-forms on AdS$_5$.

The Ansatz for F_5 was rather intricate already at first order. We write F_5 as the sum of components $F_{p,q}$ which has p legs in AdS$_5$ and q legs in X so that

$$F_5 = F_{0,5} + F_{1,4} + F_{2,3} + F_{3,2} + F_{4,1} + F_{5,0}. \tag{8.2.30}$$

[3] Forms which are closed and co-closed are automatically invariant under the isometry, hence the number of harmonic three-forms is the same as the number of invariant harmonic three-forms.

Then we take the Ansatz to be

$$F_{0,5} = \frac{2\pi N}{V}\,\mathrm{vol}_X, \qquad F_{5,0} = \frac{2\pi N}{V}\,\mathrm{vol}_{\mathrm{AdS}}, \tag{8.2.31}$$

$$F_{1,4} = \frac{2\pi N}{V} A^a \wedge \iota_{k_a}\,\mathrm{vol}_X + *_0\, F_{4,1}, \tag{8.2.32}$$

$$F_{2,3} = N F^I \wedge \omega_I, \qquad F_{3,2} = N(*F^I) \wedge *\omega_I. \tag{8.2.33}$$

Here, ω_I are three-forms on X, and F^I are two-forms on AdS_5, respectively. The range in which I can take values is $1, \ldots, d = b^3 + \ell$, as will see again. The first term in (8.2.32) is necessary because eq. (8.2.29) modifies the Hodge star.

The exterior derivative is decomposed into $d = d_X + d_{\mathrm{AdS}}$ where $d_{X,\mathrm{AdS}}$ is the exterior derivative on the respective spaces. Then, $dF_5 = 0$ imposes

$$d_{\mathrm{AdS}} F_{p,q+1} + d_X F_{p+1,q} = 0. \tag{8.2.34}$$

$F_{4,1}$ can be shown to yield massive degrees of freedom, so we set $F_{4,1} = 0$. Then, in order to have massless equation of motion $dF^I = 0$ and $d * F^I = 0$, there must be constants c_I^a such that

$$d * \omega_I = 0, \quad \text{and} \quad d\omega_I = \frac{2\pi}{V} c_I^a \iota_{k_a}\,\mathrm{vol}_X \tag{8.2.35}$$

for ω_I and

$$dA^a = c_I^a F^I \tag{8.2.36}$$

for F^I. One important thing is the non-closedness of ω_I, which was already pointed out in [18]. If $d\omega_I = 0$ in (8.2.35), the allowed number of F^I would be precisely $b^3 = \dim H^3(X)$. The presence of $\iota_{k_a}\,\mathrm{vol}_X$ enlarges the dimension of the space of wavefunctions ω_I for massless gauge fields by the number of isometries, ℓ. Thus, the index I runs from 1 to d where

$$d = \ell + b^3. \tag{8.2.37}$$

Let us introduce $\mathrm{vol}^\circ \equiv \mathrm{vol}/V$ and $k_I \equiv 2\pi c_I^a k_a$. Eq. (8.2.35) becomes

$$d\omega_I + \iota_{k_I}\,\mathrm{vol}_X^\circ = 0. \tag{8.2.38}$$

This form will be referenced in later sections.

One contribution to the Chern-Simons interaction arises as follows. The Hodge star $*$ for the metric ansatz (8.2.29) forces F_5 to have a second-order contribution of the form

$$\delta^{(2)} F \propto A^a \wedge F^I \wedge \iota_{k_a}\omega_I, \tag{8.2.39}$$

just as we had $A^a \wedge \iota_{k_a}\,\mathrm{vol}_X$ term in (8.2.32). Then, $d_{\mathrm{AdS}} F_{3,2} + d_X F_{4,1} = 0$ requires the presence of $F^a \wedge F^I$ terms on the right hand side of the equation of motion. Full analysis was carried out in the previous subsection, with the result (8.2.25). In the present notation, it becomes

$$d * F^I \int_X (\omega_K \wedge *\omega_I + \frac{1}{16V^2}\langle k_K k_I\rangle\,\mathrm{vol}) = \frac{1}{8\pi} F^I \wedge F^J \int_X \omega_{\{I} \wedge \iota_{k_J}\omega_{K\}}. \tag{8.2.40}$$

8.2.3 Comparison to the 5d Lagrangian

Let us write down the formula for c_{IJK} and τ_{IJ}. In order to determine the combination of τ_{IJ} and c_{IJK} entering the five-dimensional action, we need the normalization of the kinetic term of F_5 entering the ten-dimensional action. The self-duality of F_5 makes the problem rather subtle. As for our case, we can give up the full ten-dimensional covariance and only retain the invariance under $SO(5,1)\times SO(5)$ local Lorentz transformation. Then, the action for F_5 in (2.1.2) can be written down to be

$$S_{F_5} = \frac{1}{4\pi}\int_{\mathrm{AdS}\times X} \mathcal{F}_5 \wedge *\mathcal{F}_5 \tag{8.2.41}$$

where $\mathcal{F}_5 = F_{0,5} + F_{1,4} + F_{2,3}$.

Plugging (8.2.29) and (8.2.33) into the action above, we obtain

$$\tau_{IJ} = \frac{N^2}{2\pi}\int_X (\omega_J \wedge *\omega_I + \frac{1}{16V^2}\langle k_J k_I\rangle \,\mathrm{vol}) \tag{8.2.42}$$

where the first and second terms come from the kinetic terms for the five-form and the metric, respectively. Then, from (8.2.40), we obtain

$$c_{IJK} = \frac{N^2}{2}\int_X \omega_{\{I} \wedge \iota_{k_J}\omega_{K\}}. \tag{8.2.43}$$

The expression for τ_{IJ} agrees with the one presented in [18].

8.2.4 a and the volume

Before moving to the explicit evaluation of c_{IJK} for various Sasaki-Einstein manifolds, let us determine the central charge a from our formula (8.2.43), and check that it is inversely proportional to the volume. In this subsection, we assume X is not just an Einstein manifold but is Sasaki-Einstein.

Let J be the Kähler form of the cone $C(X)$ over X, and $e_r = r\partial_r$ the dilation on the cone direction. Let e be the one-form $\iota_{e_r}J$. It endows X with the structure of a contact manifold so that $\mathrm{vol}_X = e \wedge J \wedge J/2$ and $de = 2J$. The Reeb vector is ie_r.

Since X is now Sasaki-Einstein, the corresponding CFT is $\mathcal{N} = 1$ superconformal. Let the R-symmetry in the superconformal algebra be the linear combination $R^I Q_I$. Then, the central charge a is given by

$$a = \frac{9}{32}c_{IJK}R^I R^K R^K = \frac{N^2}{2}\frac{27}{16}\int \omega_R \wedge \iota_{k_R}\omega_R \tag{8.2.44}$$

where $\omega_R = R^I\omega_I$ and $k_R = R^I k_I$. It is known through the work [77] that ω_R is a multiple of $e\wedge J$. We should normalize it so that k_R is proportional to the Reeb vector, and the holomorphic three-form Ω to have charge 2 under k_R. Thus, we obtain

$$k_R = 2\pi\frac{2}{3}ie_r \tag{8.2.45}$$

because Ω scales as r^3 and the natural holomorphic one-form on the cone is re. The extra factor of 2π comes from our convention $k_I = 2\pi c_I^a k_a$ relating k_I in F_5 and the k_a in the metric ansatz.

Thus, we have

$$\omega_R = -\frac{\pi e\wedge J}{3V} \tag{8.2.46}$$

from (8.2.38). Then eq (8.2.44) becomes

$$a = \frac{N^2}{2}\frac{27}{16}\frac{4\pi^3}{27}\frac{\int e \wedge J \wedge J}{V^2} = \frac{N^2}{4}\frac{\pi^3}{V}, \tag{8.2.47}$$

which is precisely the relation established in [88, 56].

8.3 Properties of the supergravity formula

8.3.1 Giant Gravitons and the normalization of ω_I

We have found so far the formula for the CS coefficient c_{IJK} given in terms of three-forms ω_I on the Einstein manifold X. The gauge field in the AdS space has these forms as the wavefunction. In order to compare the result to the field theory in four dimensions, first we need to find the basis of the gauge fields so that charged objects have integral charges with respect to these gauge fields.

Let us recall the situation in the compactification of the M-theory on a Calabi-Yau Y. In that case, a massless gauge field arises from the M-theory three-form with a harmonic two-form ω on Y as the wavefunction, and harmonic two-form naturally corresponds to $H^2(Y, \mathbb{R})$. M2-branes wrapped on a two-cycle C in the Calabi-Yau give rise to the charged particles in the noncompact dimensions, and the charge is given by $\int_C \omega$. Thus, $H^2(Y, \mathbb{Z}) \subset H^2(Y, \mathbb{R})$ gives the integral basis we wanted.

Similarly in our case, D3-branes wrapped on three-cycles in the Einstein manifold X give rise to charged objects on the AdS side[4]. There are $b^3(X)$ homologically independent three-cycles. We also have ℓ Kaluza-Klein angular momenta associated to the ℓ isometry. For example, gravitons moving inside X will give charged objects from the AdS point of view. In all, there are $d = b^3(X) + \ell$ types of charged objects which match with the number of the massless gauge fields.

Let us give a simple argument showing that ordinary homology of 3-cycles is not the correct mathematical object to classify the charges of the supersymmetric wrapped D3-branes. For S^5 the homology is trivial but there are giant gravitons. A less simple example comes from the $Y^{p,q}$ geometries: there are various supersymmetric 3-cycles which are homologically equivalent but have different volumes. D3-branes wrapped on different cycles correspond to different operators in the dual quiver gauge theory. These supersymmetric 3-cycles are invariant under the $U(1)^l = U(1)^3$ isometries. The point is that we cannot deform one such supersymmetric 3-cycle to another keeping it invariant under the isometries. It is thus clear that we need some kind of homology that keeps track also of the isometries, which show up in AdS_5 as Kaluza-Klein momenta.

Another important fact is that the wavefunctions ω_I are not closed in general. Then the charge of a wrapped D3-brane depends not only on its homology class, but also on extra data, as expected also from the discussion in the previous paragraph. The Kaluza-Klein gauge fields coming from the metric also enter the expansion of F_5, because in the expansion (8.2.33)

$$\delta F_5 = d(A^I \wedge N\omega_I), \tag{8.3.1}$$

A_I includes the gauge fields from the metric through (8.2.36). The non-closedness of ω_I allows a D3-brane wrapping a topologically trivial cycle C to have a non-zero coupling to A^I given by

$$N \int_C \omega_I. \tag{8.3.2}$$

[4]The R-charge of the wrapped D3-branes was studied in [77]. The analysis of the R-charge and the baryonic charges in the regular Sasaki-Einstein manifolds was carried out in detail in [89].

For instance, if we consider the type IIB theory on S^5 with N units of five-form flux and we wrap a D3-brane on S^3 at the equator, it will give rise to a soliton with N unit of Kaluza-Klein momenta. This is precisely the maximal giant gravitons treated in [65, 66]. The point is that, although the D3-brane sits in the equator apparently at rest, it couples to the gauge field corresponding to the rotation through the mixing of the gauge fields coming from the metric and the F_5.

For simplicity, let us restrict our attention to branes which are apparently at rest in the SE. In order for them to be charge eigenstates, their worldvolume should be invariant under the isometry. Let us introduce an equivalence relation such that $C \sim C'$ if $C - C' = \partial B$ where B is an invariant four-chain. Then, the coupling of the branes to the gauge fields A^I depends only on the equivalence class, because

$$\int_C \omega_I - \int_{C'} \omega_I = \int_{\partial B} \omega_I = \int_B d\omega_I = \int_B \iota_{k_I} \operatorname{vol}^\circ, \tag{8.3.3}$$

and the integral of ι_k acting on anything vanishes if the integration region is invariant under k. It is because the integrand is zero when k degenerates on C and the interior product kills the legs along C when k does not degenerate on C.

Suppose X has $U(1)^\ell$ isometry and the third Betti number to be b^3. In the explicit examples we will treat in the following sections, there are always $d = \ell + b^3$ of independent invariant three-cycles, although we could not find a general proof in the mathematical literature[5]. Assuming this, D3-branes wrapping on invariant three-cycles form a good basis of charged objects with respect to the gauge fields A^I. Let us denote the basis by C^I, $(I = 1, \ldots, d)$. Then, the conditions

$$\int_{C^I} \omega_J = \delta^I_J, \tag{8.3.4}$$

determines the dual basis for the wavefunctions of the gauge fields A_I. Then a D3-brane wrapping the cycle C^I has charge N under A_I, and charge 0 for other gauge fields.

8.3.2 Metric independence of c_{IJK}

In sec. 8.2, the form ω_I is co-closed and 'closed up to isometry' (8.2.38). We show in this section that c_{IJK} and the normalization condition do not change when ω_I and vol$^\circ$ are shifted by exact forms $d\alpha$ where α is invariant under the isometries. Thus, the knowledge of the metric is not required in the calculation of c_{IJK}.

First, we can freely shift ω_I by exact forms, $\omega_I \to \omega_I + d\alpha_J$, without affecting c_{IJK} and the normalization condition (8.3.4). The latter statement is obvious. The former one can be verified easily as follows:

$$\delta c_{IJK} \propto \int d\alpha_{\{I} \wedge \iota_{k_J} \omega_{K\}} = -\int \alpha_{\{I} \wedge \iota_{k_J} d\omega_{K\}} = -\int \alpha_{\{I} \wedge \iota_{k_J} \iota_{k_{K\}}} \operatorname{vol}^\circ_X = 0. \tag{8.3.5}$$

Secondly, we can add exact forms $d\alpha$ to $\operatorname{vol}^\circ_X$ without affecting c_{IJK} nor the condition (8.3.4). The change induces the change $\omega_I \to \omega_I + \iota_{k_I}\alpha$ via (8.2.38), which in turn causes c_{IJK} to change by

$$\delta c_{IJK} = \int \iota_{k_{\{I}} \alpha \wedge \iota_{k_J} \omega_{K\}} = 0. \tag{8.3.6}$$

[5] In [67, 68], one can find interesting discussions on the construction of the supersymmetric three-cycles using the complex algebraic geometry of the cone over the Sasaki-Einstein manifolds.

Hence it does not change the CS coefficient. As for the normalization (8.3.4), the cycles C^I are assumed to be invariant under the isometry. Then we have $\int_{C^I} \iota_{k_J}\alpha = 0$, using the same argument as before.

Let us recapitulate the method to calculate c_{IJK}:

- We first take any invariant five-form vol° which satisfies $\int \mathrm{vol}^\circ = 1$.
- Then find ω_I with the normalization $\int_{C^J} \omega_I = \delta_I^J$, (8.3.4).
- Next we define k_I as the linear combination of ℓ isometries such that the condition $d\omega_I + \iota_{k_I} \mathrm{vol}^\circ = 0$, (8.2.38) is satisfied.
- Finally we plug these quantities into the formula (8.2.43) and evaluate it.

The procedure does not require the knowledge of the Einstein metric on X. We would like to emphasize that the Sasaki structure on X is not necessary in the calculation of c_{IJK} either. The only ingredient is the action of $U(1)^\ell$ on X. In this sense we claim that c_{IJK} is a topological invariant of the manifold with $U(1)^\ell$ action.

8.4 Explicit Evaluation of the supergravity formula

8.4.1 Sasaki-Einstein manifolds with one $U(1)$ isometry

We first treat the case where there is only one isometry k on the Sasaki-Einstein manifold X. We take the period of k to be 2π. Then, the isometry determines on X an S^1 fibration

$$\begin{array}{ccc} S^1 & \to & X \\ & & \downarrow \\ & & B \end{array} \tag{8.4.1}$$

over a Kähler-Einstein base B. Let the one-form e be $e = g_{ij}k^i dx^j$. Then, the Sasaki-Einstein condition implies that the curvature of the circle bundle de is equal to twice the Kähler class J of the base B, that is,

$$de = 2J. \tag{8.4.2}$$

We have $\mathrm{vol}^\circ \propto e \wedge J \wedge J$. Then, an elementary calculation shows that elements of $H^3(X)$ corresponds to elements of $H^2(B)$ annihilated by $J\wedge$. Thus, $b^3(X) = b^2(X) - 1$. Since we assumed $\ell = 1$, the number of the gauge field d is

$$d = \ell + b^3(X) = b^2(B). \tag{8.4.3}$$

Thus, we need to find $b^2(B)$ of three-cycles C^I and three-forms ω_I in X which satisfy the constraint (8.2.38) and (8.3.4). To this end, take a basis of two-cycles $D^1, \ldots, D^d$ in B and the dual basis of two-forms $\gamma_1, \ldots, \gamma_d$ on B such that $\int_{D^I} \gamma_J = \delta^I_J$. Let us take C^I to be the three-cycle above D^I in the fibration and $\omega_I = (2\pi)^{-1} e \wedge \gamma_I$. Then the normalization (8.3.4) is automatic, and from $d\omega_I + \iota_{k_I} \mathrm{vol}^\circ = 0$ (8.2.38), we have

$$k_I = -2(\int_B J \wedge \gamma_I)k. \tag{8.4.4}$$

Thus we obtain

$$c_{IJK} = \frac{N^2}{2} \int_B \frac{J}{\pi} \wedge \gamma_{\{I} \int_B \gamma_J \wedge \gamma_{K\}}. \tag{8.4.5}$$

8.4.2 Higher del Pezzo surfaces

Circle bundles over del Pezzo surfaces are prime examples of five-dimensional Sasaki-Einstein manifolds, where the n-th del Pezzo surface dP_n for $n < 9$ is $\mathbb{CP}^2$ blown up at generic n points. For $n = 1, 2, 3$ they are toric, which will be treated in the next subsection. In this subsection we evaluate (8.4.5) for del Pezzo surfaces with $n \geq 4$, which have only one isometry which rotates the circle fiber. We compare the result with the field theory result in section 8.5.1.

Let us take γ_0 as the two-form dual to the base $\mathbb{CP}^2$, and γ_i, $i = 1, \ldots, n$ be the two-forms dual to the i-th exceptional cycle. The intersection paring is Lorentzian, i.e.

$$\int_{\mathrm{dP}_n} \gamma_I \wedge \gamma_J = \mathrm{diag}(+1, -1, \ldots, -1). \tag{8.4.6}$$

where $I, J = 0, 1, \ldots, n$. The Kähler form J is chosen to be equal to the negative of the Chern class of the anti-canonical bundle,

$$J = \frac{\pi}{3}(3\gamma_0 - \sum_{i=1}^{n} \gamma_i). \tag{8.4.7}$$

The area of the dP_n is then $\int_{\mathrm{dP}_n} J \wedge J/2 = \pi^2(9 - n)/18$. Formula (8.4.5) can be conveniently packed in the cubic polynomial

$$P_n(a_0, a_1, \ldots, a_n) \equiv c_{IJK} a^I a^J a^K = 3N^2 \int_{\mathrm{dP}_n} \frac{J}{\pi} \wedge \gamma \int_{\mathrm{dP}_n} \gamma \wedge \gamma \tag{8.4.8}$$

by introducing indeterminate variables a^I, $I = 0, \ldots, n$ and $\gamma \equiv \gamma_I a^I$. It can be easily evaluated to be

$$P_n(a^I) = N^2 \left(3a^0 + \sum_i a^i\right)\left((a^0)^2 - \sum_i (a^i)^2\right). \tag{8.4.9}$$

An obvious consequence is that we have

$$P_n(a^0, a^1, \ldots, a^n) = P_{n+1}(a^0, a^1, \cdots, a^n, a^{n+1} = 0). \tag{8.4.10}$$

We will see the physical mechanism behind it in later sections.

8.4.3 Toric Sasaki-Einstein manifolds

We would like to move on to the case where there are three isometries in the Sasaki-Einstein manifold X, i.e. $\ell = 3$. In that case, the Calabi-Yau cone over X is toric, thus X is called a toric Sasaki-Einstein manifold. Let us describe X as a T^3 fibration over a two-dimensional d-gon B, where the coordinates of T^3 are $\theta_{1,2,3}$ and those of the base are $y^{1,2}$. We take the periodicity of θ_i to be 1. Denote the edges by E^I, $I = 1, \ldots, d$, the 3-cycles above them by C^I. It is known that $H^3(X) = d - 3$ so that the number of the edges is precisely the number of gauge fields which we obtain by compactifying the type IIB string on X. Let $k_I = k_{iI}\partial/\partial\theta_i$ be the degenerating Killing vector at C^I, see figure 8.1.

We will see shortly that the calculation of c_{IJK} only depends on $k_{I,J,K}$ and not on the other $k_{L\neq I,J,K}$ or the number of the edges. From now on, all the forms are assumed to depend only on $y^{1,2}$.

Firstly, take a two-form $\mathcal{F}$ on the base B supported on a region S with $\int \mathcal{F} = 1$. S is marked with dark grey in the figure 8.1. Choose

$$\mathrm{vol}^\circ = \mathcal{F} \wedge d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \tag{8.4.11}$$

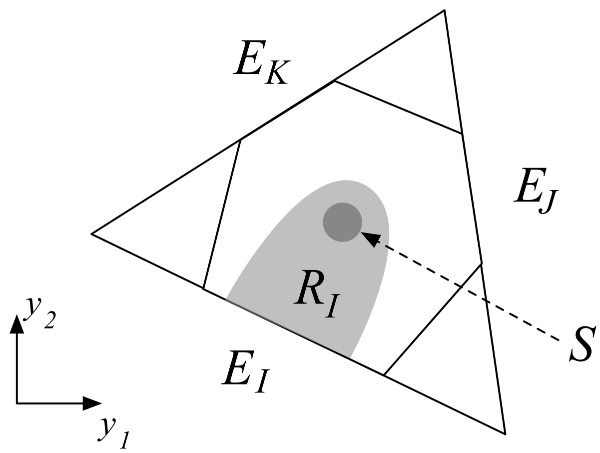

Figure 8.1: Construction of ω_I. The polygon designates the image of the moment map. The dark grey blob S is the support of $\mathcal{F}$ and the pale grey region R_I is the support of $\mathcal{A}_I$.

as the normalized volume form.

Secondly, for each edge E_I, draw a region R_I which contains S and touches only with E_J with $J = I$ (cf. fig. 8.1). Choose a one-form $\mathcal{A}_I$ on the base B which is non-zero only in R_I such that $d\mathcal{A}_I = \mathcal{F}$. Notice $\sum_J \int_{E_J} \mathcal{A}_I = \int_B \mathcal{F} = 1$, $\int_{E^J} \mathcal{A}_I = 0$ for $J \neq I$ thus $\int_{E^I} \mathcal{A}_J = \delta^I_J$.

We need to ensure furthermore that $\mathcal{A}_I$ has only components parallel to the edge E_I. Then

$$\omega_I \equiv -\mathcal{A}_I \wedge \iota_{k_I} d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \qquad \text{(no summation on } I) \tag{8.4.12}$$

is a well-behaved form on X, since the existence of ι_{k_I} guarantees that ω_I is regular near E_I, and the fact A_I vanishes outside the pale grey region R_I guarantees ω_I is regular near $E_{J\neq I}$. It also satisfies the constraint (8.2.38) and (8.3.4) almost by construction.

Now we can clearly see that the forms $\omega_{I,J,K}$ can be taken to be the same irrespectively of, for example, whether we are calculating c_{IJK} for the hexagon inside or the triangle outside in the figure. Thus, c_{IJK} depends only on $k_{I,J,K}$ and not at all on $k_{L\neq I,J,K}$. It is even independent of the number of the edges, i.e.

$$c_{IJK} = f(k_I, k_J, k_K) \tag{8.4.13}$$

for some function f.

First of all, if two of $k_{I,J,K}$ are equal, then f is obviously zero because the integrand is zero. Next, let us consider the case when they are all different. We can assume the base B is a triangle without loss of generality. We will show that X is an orbifold of S^5, which allows us to obtain c_{IJK}.

Take the universal cover U of X, that is, remove the periodicity $\theta_i \sim \theta_i + 1$. X can be obtained by dividing S^5 with the lattice N generated by $(\theta_1, \theta_2, \theta_3) = (1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Instead, consider a manifold Y by dividing U by the lattice L generated by k_I, k_J and k_K. Along the edges of B, precisely the direction $k_{I,J,K}$ degenerates. Thus we have shown that Y is topologically an S^5, and $X = S^5/\Gamma$ where Γ is the finite group L/N. The order of Γ is

$$\#\Gamma = |\det(k_I, k_J, k_K)|\,. \tag{8.4.14}$$

Let us denote the corresponding quantities on S^5 by adding tildes and the projection map by $i : S^5 \to S^5/\Gamma = X$, we find

$$i^*\omega_I = (\#\Gamma)\tilde{\omega}_I, \quad i^*\,\mathrm{vol}^\circ = (\#\Gamma)\,\widetilde{\mathrm{vol}}^\circ, \quad \text{and} \quad i^*k_I = \tilde{k}_I. \tag{8.4.15}$$

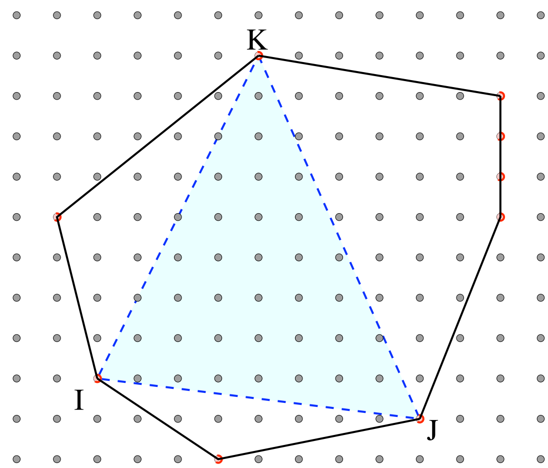

Figure 8.2: Pictorial representation of the toric formula $c_{IJK} = \frac{N^2}{2}\,|\det(k_I, k_J, k_K)|$.

Then

$$\int_{S^5/\Gamma} \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}} = (\#\Gamma)^{-1} \int_{S^5} i^* \omega_{\{I} \wedge \iota_{k_J} i^* \omega_{K\}} = \#\Gamma \int_{S^5} \tilde{\omega}_{\{I} \wedge \iota_{\tilde{k}_J} \tilde{\omega}_{K\}} \tag{8.4.16}$$

that is, c_{IJK} is $\#\Gamma$ times that of S^5. Finally, for S^5, one can do the explicit calculation to find $c_{IJK} = N^2/2$. Thus we obtain the formula

$$c_{IJK} = \frac{N^2}{2}\,|\det(k_I, k_J, k_K)|\,, \tag{8.4.17}$$

which is proportional to the area of the triangle inside the toric diagram, see figure 8.2.

8.5 Triangle Anomalies for corresponding quiver theories

8.5.1 del Pezzo surfaces

Now we want to discuss the gauge theories corresponding to the complex cones over del Pezzo surfaces. The quivers were constructed in [69] for toric del Pezzo surfaces (dP_1, dP_2 and dP_3), and in [78, 90] for the non toric ones, i.e. dP_n with $4 \le n \le 8$. The generic superpotential for dP_5 and dP_6 was derived in [91], for dP_7 and dP_8 the explicit superpotential is still not known. In [92, 93], all the baryonic and R charges are explicitly listed for dP_n up to $n = 6$. It is simple to compute, using these data, the cubic 't Hooft anomalies and to match with our geometrical findings in sec. 8.4.2.

In [89], the R- and baryonic charges of the dibaryons were analyzed through the framework of the exceptional collections on the del Pezzo surfaces. In particular, they showed that the triangle anomaly among one R-symmetry and two baryonic symmetries $\mathrm{tr}(RB_1B_2)$ is proportional to the intersection form of the two-cycles which are perpendicular to the Kähler class of the surface. It is easy to check that our formula in sec. 8.4.2 naturally reproduces their result.

8.5.2 Toric cones with four edges

Let us report in detail the results for the case of toric diagram with 4 corners. The toric diagram is given in figure 8.3 and the global symmetry charges are given by table 8.1. The charge assignments is

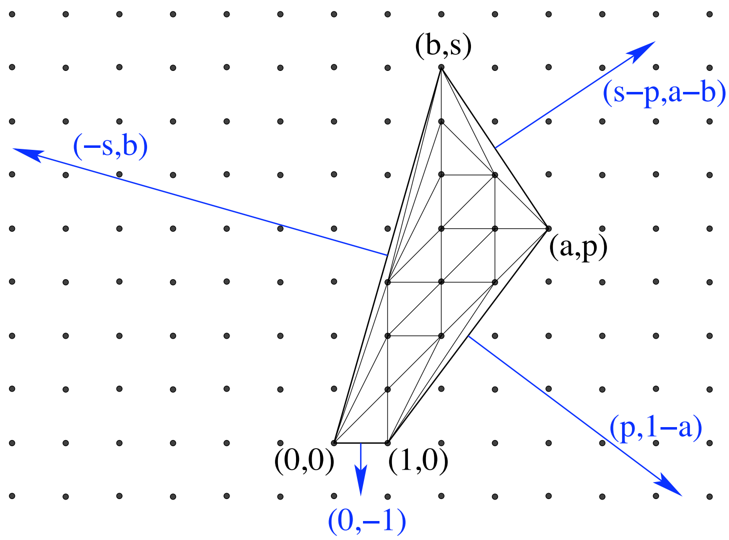

Figure 8.3: A generic toric diagram with four corners, i.e. a generic $L^{p,q|r}$, and the associated (p,q)-web. We have $s = p + q - r$. The integers a and b are such that $as - bp = q$.

Field	Number	Q_1^B	Q_2^B	Q_3^B	Q_4^B	Q_1^F	Q_2^F	Q_3^F	Q_4^F
$\mathcal{B}_{12}$	p	1	0	0	0	1/2	−1/2	−1/2	−1/2
$\mathcal{B}_{23}$	r	0	1	0	0	−1/2	1/2	−1/2	−1/2
$\mathcal{B}_{34}$	q	0	0	1	0	−1/2	−1/2	1/2	−1/2
$\mathcal{B}_{41}$	$p+q-r$	0	0	0	1	−1/2	−1/2	−1/2	1/2
$\mathcal{B}_{13}$	$q-r$	1	1	0	0	1/2	1/2	−1/2	−1/2
$\mathcal{B}_{42}$	$r-p$	1	0	0	1	1/2	−1/2	−1/2	1/2
Gauge	$p+q$	0	0	0	0	1/2	1/2	1/2	1/2

Table 8.1: Charge assignments for the basic superfields in the case of toric diagrams with four corners.

determined from the rule explained in section 7.3. The column labeled by Q_I^B lists the charge of the bosonic components under the global symmetry generator Q_I, and that labeled by Q_I^F is the charge of the fermion in the multiplet.

It is straightforward to check that the linear 't Hooft anomalies vanish, i.e. $\mathrm{tr}(Q_J) = 0$. This has to be the case for any superconformal quiver [92, 94]. A general proof of the vanishing of linear anomalies using the folded quiver picture was given in [17]. Since $(Q_I^F)^2 = 1/4$, $\mathrm{tr}(Q_J) = 0$ also implies that

$$\mathrm{tr}(Q_I^2 Q_J) = \mathrm{tr}(Q_J) = 0 \tag{8.5.1}$$

The remaining cubic 't Hooft anomalies (recall they are completely symmetric) are easily computed to be

$$\begin{aligned}
\mathrm{tr}(Q_1Q_2Q_3) &= N^2\, r/2 && (8.5.2)\\
\mathrm{tr}(Q_2Q_3Q_4) &= N^2\, q/2 && (8.5.3)\\
\mathrm{tr}(Q_3Q_4Q_1) &= N^2 (p+q-r)/2 && (8.5.4)\\
\mathrm{tr}(Q_4Q_1Q_2) &= N^2\, p/2 && (8.5.5)
\end{aligned}$$

It is now straightforward to check that these are proportional to the area of the triangles

$$|\det(k_I, k_J, k_K)| \tag{8.5.6}$$

spanned by the corners of the toric diagram of figure 8.3. Thus we have shown that, for a toric diagram with four edges, the cubic anomaly c_{IJK} is given by

$$c_{IJK} = \frac{N^2}{2}|\det(k_I, k_J, k_K)|, \tag{8.5.7}$$

which agrees with the supergravity result (8.4.17).

This nice result can be proven for a generic toric diagram with arbitrary number of edges, by an easy mathematical induction, which is the topic of the next subsection.

8.5.3 Triangle anomaly for general toric quivers

Here we are going to prove the formula

$$c_{IJK} = \frac{N^2}{2}|\det(k_I, k_J, k_L)| \tag{8.5.8}$$

for quiver gauge theories on the D3-branes probing the tip of a toric Calabi-Yau cone. Let us denote by $k_I = (1, \vec{k}_I)$ $(I = 1, \ldots, d)$ the toric data of the toric Calabi-Yau manifold. We set $k_0 \equiv k_d$. One can express the same data using the language of the (p, q)-web, in which the direction of the i-th external leg is given by $(p_i, q_i) = \vec{k}_i - \vec{k}_{i-1}$. The field content of the corresponding quiver theory is summarized in sec. 7.3, Properties 1, 2 and 3, and the global symmetry charges are described by Properties 4 to 7. Let us consider a linear combination $Q = a^I Q_I$ of the $U(1)$ charges Q_I. Then, the charge of the superpotential under Q is $\sum a_I$ and the charge of the chiral superfields in $\mathcal{B}_{ij}$ is

$$\sum_{K=i}^{j-1} a^K = a_i + a_{i+1} + \cdots + a_{j-1}. \tag{8.5.9}$$

The number n_{ij} of chiral superfields in $\mathcal{B}_{ij}$ is given by the intersection number of the two (p, q)-legs, that is,

$$n_{ij} = \det(\vec{k}_j - \vec{k}_{j-1}, \vec{k}_i - \vec{k}_{i-1}), \tag{8.5.10}$$

while the number n_V of gauge groups is given by the area of the toric diagram

$$n_V = \sum \det(\vec{k}_I - \vec{k}_1, \vec{k}_{I+1} - \vec{k}_1). \tag{8.5.11}$$

Then the triangle anomaly among three Q's is given by

$$\frac{1}{N^2} c^{\mathrm{CFT}}_{IJK} a^I a^J a^K = n_V (\frac{1}{2}\sum a^I)^3 + \sum_{I<J} n_{IJ} \left(\sum_{K=I}^{J-1} a^K - \frac{1}{2}\sum a^I \right)^3 . \tag{8.5.12}$$

This expression follows from the folded-quiver picture of [76], and was first explicitly written down in the work of Butti and Zaffaroni [17]. In the usual formula we have 1 instead of $\sum a^I/2$; we would like to have the triangle anomaly including the global symmetry usually fixed by $\sum a^I = 2$, so we resurrected that combination.

One can show, by mathematical induction, c^{CFT}_{IJK} only depends on $k_{I,J,K}$ and not on other k_L for $L \neq I, J, K$. nor on the number of edges. The proof goes as follows :

Suppose $I, J, K \neq d$ and let us show c_{IJK} is independent of k_d. Consider two toric data, one is the original set $\{k_1, k_2, \cdots, k_d\}$ and the other is $\{k_1, \cdots, k_{d-1}\}$ without k_d. Let us distinguish various quantities for the latter by adding tilde above, e.g. $\tilde{n}_V$ and so on. Then we have two relations

$$n_{I,d-1} + n_{I,d} = \tilde{n}_{I,d-1} \tag{8.5.13}$$

and

$$n_V - n_{d-1,d} = \tilde{n}_V. \tag{8.5.14}$$

Applying them to the formula (8.5.12), we obtain

$$c^{\rm CFT}_{IJK} a^I a^J a^K \Big|_{a^N=0} = \tilde{c}^{\rm CFT}_{IJK} a^I a^J a^K. \tag{8.5.15}$$

Thus, c_{IJK} for $I, J, K \neq d$ is independent of k_d. Inductively, we can show that c_{IJK} depends only on k_I, k_J and k_K.

Hence, we can obtain $c^{\rm CFT}_{IJK}$ by considering the case of a triangle. One can easily show that, in this case,

$$n_V = n_{IJ} = n_{JK} = n_{KI} = |\det(k_I, k_J, k_K)|. \tag{8.5.16}$$

Plugging in to the formula (8.5.12), we finally obtain

$$c^{\rm CFT}_{IJK} = \frac{N^2}{2} |\det(k_I, k_J, k_K)|. \tag{8.5.17}$$

It precisely agrees with the result from the supergravity analysis (8.4.17).

8.6 a-maximization vs. Z-minimization

In the previous section we established the formula for the triangle anomaly c_{IJK} in terms of the toric data of the cone. On one hand, one can form trial a-function from c_{IJK} and then to carry out the a-maximization. On the other hand, one can carry out the Z-minimization process explained in sec. 6.5 directly to the toric cone. From the AdS/CFT correspondence, a at the maximum and Z at the minimum should be inversely proportional with the proportionality constant specified in (6.2.10). Thus it is a natural question whether the relation between a and Z can be established directly without recourse to AdS/CFT correspondence. When done, it will give another generic test of the correspondence.

There is an apparent difficulty in the matching of a and Z. First of all, the number of variables for the trial a function is $d-1$ with d the number of edges, while for the trial Z function it is always two, irrespective of the number of edges. Furthermore, the function a is cubic, while the function Z is a complicated rational function. With all the difficulties, the authors of [17] established a miraculous identity. To present it, we need to introduce some notation to make it easier to understand. Let

$$L^I(R) = \frac{\det(k_{I-1}, k_I, k_{I+1})}{\det(R, k_{I-1}, k_I)\det(R, k_I, k_{I+1})}, \tag{8.6.1}$$

and $S(R) = \sum L^I(R)$. The inverse of the Z function (6.5.12), $a_{MSY}(R) \propto 1/Z(R)$, is given by

$$a_{MSY}(R) = \frac{9}{32} \frac{24}{S(R)}, \tag{8.6.2}$$

where it is normalized so that we expect it to match as $a_{MSY}(R) = a_{CFT}$ at its maximum. We drop the overall coefficient N^2 for brevity below. Then the relation found and proved in [17] is

$$a_{MSY}(R) = a_{CFT}(a^I = \frac{2L^I(R)}{S(R)}), \tag{8.6.3}$$

even without the maximization with respect to R.

The proof in [17] was extremely long, which was dramatically simplified with the use of the formula (8.5.8) in [81]. We present the proof following the latter reference. Let $R = (1, \vec{r})$. Firstly, note that $L^I(R - k_I)$ with no sum in I satisfies

$$L^I(R - k_I) = \frac{k_I - k_{I-1}}{\det(R, k_{I-1}, k_I)} - \frac{k_{I+1} - k_I}{\det(R, k_I, k_{I+1})} \qquad \text{(no sum on } I\text{)}. \tag{8.6.4}$$

Summing over I, we get

$$\frac{L^I}{S} k_I = R. \tag{8.6.5}$$

Another key is the lemma that the relation

$$L_I \equiv c_{IJK} L^J L^K = 3S + \det(\vec{r} - \vec{k}_I, \vec{u}) \tag{8.6.6}$$

holds with a fixed two-column vector $\vec{u}$. The proof is postponed to the end of this section.

Combination of (8.6.5) and (8.6.6) immediately implies

$$L_I B^I = c_{IJK} L^I L^J B^K = 0 \tag{8.6.7}$$

for baryonic symmetries B^I, which satisfies $B^I k_I = 0$ by definition. Additionally, the relation

$$a_{CFT}(a^I = \frac{2L^I}{S}) = \frac{9}{32}\frac{8 L_I L^I}{S^3} = \frac{9}{32}\frac{24}{S} = a_{MSY}(R) \tag{8.6.8}$$

also follows easily, which was to be shown.

Before discussing the proof of the lemma, we would like to study the physical significance of the relations (8.6.5), (8.6.7) and (8.6.8). Recall that for charges $a^I Q_I$, its projection to the mesonic charges modulo the baryonic charges is found by forming the linear combination $a^I k_I$. Then the equation (8.6.5) means that the charges $2L^I(R)Q_I/S(R)$ is a particular combination of the global symmetries with the given mesonic charge $2R$. The relative coefficient 2 comes from our normalization in this chapter that $\sum a^I = 2$. The combination is fixed uniquely by (8.6.7), that is,

$$\frac{\partial}{\partial B^I} a_{CFT}\Big|_{a_I = 2L^I/S} = 0. \tag{8.6.9}$$

Thus, the charges on the left hand side of (8.6.8)

$$a^I(R) = \frac{2L^I(R)}{S(R)} \tag{8.6.10}$$

is chosen so that

- $a^I(R) k_I = 2R$, and
- it maximize the trial a-function keeping $a^I(R) k_I$ fixed.

In other words, $a^I(R)$ is obtained by partially maximizing the a-function along the baryonic direction. Then the result (8.6.8) means that, after the partial maximization just discussed, the Z function is inversely proportional to the CFT a function.

What is surprising here is that the relation (8.6.8) holds without the maximization over R. Indeed, the AdS/CFT relation (6.2.10) is expected to hold only at the superconformal point. (8.6.8) implies that the relation (6.2.10) still holds slightly off-shell in the sense of five-dimensional supergravity. It might be expected that we can find a physical explanation for (8.6.8) through the dimensional reduction for the scalar potential in five-dimensions, as was done for gauge fields and their Chern-Simons interactions in section 8.2. It is because the Z function is related to the volume of X^5, which in turn is linked to the vacuum energy in the five-dimensional theory in AdS. It would be very interesting to carry out this program.

Let us show the lemma to conclude this section. Define

$$\vec{u}_I = \sum_{I+1\le J<K\le I+d} (\vec{r}_J - \vec{r}_K)L^J L^K - 2S\,\frac{\vec{k}_I - \vec{k}_{I+1}}{\det(R, k_I, k_{I+1})}. \tag{8.6.11}$$

One can show $\vec{u}_I = \vec{u}_{I+1}$ using (8.6.5). Thus we can drop the index I from u_I from now on. Then $D_I \equiv c_I - \det(\vec{r} - \vec{k}_I, \vec{u})$ is also independent of I. Using (8.6.5) again, we obtain

$$D_1 = 2S + \sum_{2\le J<K\le d} \det(\vec{r} - \vec{k}_J, \vec{r} - \vec{k}_K)L^I L^K. \tag{8.6.12}$$

The second term in the right hand side is equal to S, which can be proven by the induction in the number of the edges d. It implies

$$c_I = 3S + \det(\vec{r} - \vec{k}_I, \vec{u}), \tag{8.6.13}$$

which was the lemma to be shown.

8.7 Rolling among Sasaki-Einstein vacua

The triangle anomalies on the CFT side and the Chern-Simons coefficients on the gravity side showed a remarkable behavior. Namely, for quiver theories for toric Sasaki-Einstein manifolds, the coefficient c_{IJK} is determined solely by the toric data $k_{I,J,K}$ and is independent of other k_L for $L \neq I, J, K$ (8.4.13). We would like to give a heuristic physical interpretation of this fact. The same consideration can be applied to the del Pezzo cases, and its manifestation is (8.4.10). We concentrate on the toric cases below.

Consider a toric Sasaki-Einstein X whose dual toric diagram has d edges. Each edge E_I naturally corresponds to a global symmetry Q_I in the quiver theory. There are bifundamental fields Φ^I with charge δ^I_J under Q_J. Then, we can form a dibaryon operator

$$B^I = \epsilon_{i_1 i_2 \ldots i_N}\epsilon^{j_1 j_2 \ldots j_N}\Phi^{I i_1}{}_{j_1}\Phi^{I i_2}{}_{j_2}\cdots\Phi^{I i_N}{}_{j_N}. \tag{8.7.1}$$

It has the charge $N\delta^I_J$ under Q_J, which is precisely the charge (8.3.2) of a D3-brane wrapping the three-cycle determined by E_I.

Now, let us give a vacuum expectation value (vev) to B_I. Since B_I is charged only with respect to Q_I and not to $Q_{J\neq I}$, the theory flow to a theory with $d-1$ global symmetries. On the gravity side, the Higgsing means that D3-branes wrapping around C_I is condensed, which presumably shrinks it just as in the blackhole condensation [95], see figure 8.4. It is the blow-down of the toric divisor corresponding to E_I on the Calabi-Yau cone over X. This procedure was used in the determination of the del Pezzo quiver in [90].

Recall that the same triangle anomaly can be calculated either in the ultraviolet or in the infrared. Thus, the triangle anomaly c_{JKL} among the global symmetries other than Q_I is the same before and after the Higgsing. Since the Higgsing eliminates the edge E_I, this means that c_{JKL} is independent

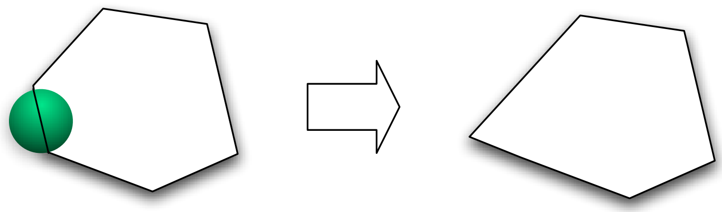

Figure 8.4: Schematic depiction of the dibaryon condensation. Each edge corresponds to a three-cycle in the toric Sasaki-Einstein around which D3-branes can be wrapped. Higgsing with the corresponding dibaryon operator in the quiver CFT eliminates that edge.

of k_I. One can repeat the flow as many times as one likes, and we can reduce the toric diagram to a triangle, which is an orbifold of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory.

Let us consider the behavior of the central charge a along the flow. Consider a flow from the UV quiver theory to the IR quiver theory triggered by giving a vev to B_I. The IR theory contains also a free chiral scalar field which represents the fluctuation of the vev of B_I. Its contribution to a is of order $1/N^2$ compared to the contribution from the interacting part, so we can neglect them henceforth. Then, from the invariance of c_{IJK} along the flow (8.4.13), the central charge a in the IR theory can be obtained by maximizing the same function as that for the UV theory in a smaller region. Thus, a will presumably decrease, with the usual caveat on the fact that the trial function attains the maximum only locally.

Let us compare the process we saw in this section with the rolling among Calabi-Yau vacua [96]. There, theories on various topologically-distinct Calabi-Yau manifolds are connected by adiabatically changing the moduli. Here, theories on various topologically-distinct Sasaki-Einstein manifolds are connected by the renormalization-group flow induced by the Higgsing of the dibaryons. Both have the same number of supercharges, and both can be understood as the Higgsing. Thus, we suggest to dub the phenomenon we found as the "rolling among Sasaki-Einstein vacua," although the rolling is unidirectional. More detailed analysis of the rolling is clearly necessary and will be interesting.

8.8 More on the charge lattice

We would like to elaborate on the mathematics of the structure of the charges of the D3-branes[6]. The case for the toric Sasaki-Einstein manifolds were analyzed in ref. [97] mainly from the point of view of the toric geometry of the cone. We discuss the problem for arbitrary Einstein manifolds.

Let us denote the space of Killing vectors by N, which can be identified with the Lie algebra of $U(1)^\ell$. It comes with a natural integral structure by stating that $k \in N$ is one of the lattice points if and only if $e^{2\pi k} = id$. Denote the dual space of N by M. Integral points of M correspond to representations of $U(1)^\ell$. The Reeb vector $R \in N$ is given when we endow X with the Sasaki structure. If X is Sasaki-Einstein, all the toric data $k \in N$ should be on a plane. It is given by a distinguished element $P \in M$ as $\langle P, k\rangle = 1$.

We deliberately used the letters M and N to evoke the connection with the toric geometry. Indeed they are precisely M and N lattices of the cone over X, if $\ell = 3$.

[6]The same analysis can be done for $(d-2)$-branes wrapping $(d-2)$-cycles in a d-dimensional manifold with isometry, since the mixing of the gauge fields coming from the metric and form-fields is a generic feature independent of the self-duality of the form-field, see [18]. We would like to thank A. Neitzke for raising this question.

We only consider the branes which wrap three-cycles invariant under the action of $U(1)^\ell$. As discussed in section 8.3.1, two cycles are taken to be equivalent if they form the boundaries of an invariant four-chain. Let us call the group of the equivalence classes of such three-cycles as $HG_3(X)$ where G stands for Giant Gravitons. We also denote the space of linear combinations of ω_I by $HG^3(X)$, where ω_I are closed up to isometry (8.2.38).

We have an exact sequence

$$0 \to H^3(X) \to HG^3(X) \to N \to 0 \tag{8.8.1}$$

where the second arrow is just the inclusion, and the third arrow is given by (8.2.38). The exactness of the sequence is also obvious.

Correspondingly, we also have another exact sequence

$$0 \to M \xrightarrow{\iota} HG_3(X) \xrightarrow{\pi} H_3(X) \to 0 \tag{8.8.2}$$

where we assumed, as before, that we can take an invariant representative for all $H_3(X)$. Then, the third arrow π is just loosening of the equivalence relation. The second arrow ι is a bit tricky to define, so we postpone the discussion to the end of this section. In the toric case, the above sequence can be obtained from the usual sequence [63]

$$0 \to M \to \mathrm{Div}_T(C(X)) \to \mathrm{Pic}(C(X)) \to 0. \tag{8.8.3}$$

for the cone $C(X)$ over X, where Div_T denotes the group of toric divisors and Pic is the Picard group.

Two exact sequences have a clear physical interpretation. First, the relation between various gauge fields are given by (8.8.1). $H^3(X)$ is the wavefunction for the purely 'baryonic' gauge fields, i.e. gauge fields coming from F_5. The elements of N are the Killing vector fields of X, which give rise to the metric Kaluza-Klein gauge field. Formula (8.8.1) says that the total space of the gauge field is given by combining the metric and F_5 gauge fields, and that there is generally no gauge fields which come purely from the metric.

Secondly, the sequence (8.8.2) relates various charges. Namely, M measures the Kaluza-Klein angular momenta, and $H_3(X)$ measures the D3-brane charges wrapping various cycles. The fact that $HG_3(X)$ is the extension of $H_3(X)$ by M tells us that, although we can have excitations with purely Kaluza-Klein momenta and without D-brane charges, e.g. gravitons, generically any states with D-brane i.e. 'baryonic' charges also have angular momenta. It also matches nicely with the result in the recent works [98, 99] which studied the BPS states with no baryonic charges and their charge lattice through the analysis of the spectrum of the Laplacian. The states without D-brane charges also appear as the semiclassical strings moving along the null geodesics. The analysis for $Y^{p,q}$ was carried out in ref. [73].

In the literature on the Sasaki-Einstein/Quiver duality, relatively little attention is paid to the M part of the charges and the N part of the gauge fields, so it seems worthwhile to study further.

Let us now come back to the construction of the second arrow ι in (8.8.2). Take an integral basis of Killing vectors v_a, $a = 1, \ldots, \ell$ of N and take the dual basis u^a in M. The basic idea is first to remove subsets X_a from X so that $X \setminus X^a$ has a trivial S^1 bundle structure under the action of the vector field v_a, second to take a section of the bundle with its graph Z_a, and finally to set $\iota(v_a) \equiv \partial Z_a$.

The bundle structure is non-trivial, thus one cannot take a genuine section. The best one can do is to get a four-chain. Then, the boundary of the four-chain is the desired image under ι. To construct an element $\iota(u^a)$ in $HG_3(X)$ for u^a, first let us denote by Y^a the three-cycle where the Killing vector

v_a degenerates. Define $B^a = (X \setminus Y^a)/U(1)_a$ where $U(1)_a$ is generated by v_a. Then, the orbit of v_a determines a genuine S^1 bundle

$$S^1 \to X \setminus Y^a \xrightarrow{p} B^a. \tag{8.8.4}$$

Consider the associated vector bundle over B^a obtained by the fiber S^1 by $\mathbb{C}$, and take a generic section of it. Let the zero locus of the section be given by $t^{ai}\gamma_i^a$ where γ_i^a is a two-dimensional submanifold of b^a and t^{ai} is the multiplicity of the zero at γ_i^a. Then consider the bundle

$$S^1 \to X \setminus (Y^a \cup \bigcup_a p^{-1}(\gamma_i^a)) \to B^a \setminus \bigcup_a \gamma_i^a. \tag{8.8.5}$$

It is a trivial S^1 bundle because we removed γ_i^a, and we can take a section Z^a of it.

Using Z_a, we define the image of u^a by ι as

$$\iota(u^a) \equiv \partial Z^a = Y^a + t^{ai} p^{-1}(\gamma_i^a). \tag{8.8.6}$$

As before, we assume that we can take Y^a and γ_i^a to be invariant under isometries.

The exactness of the sequence (8.8.2) is now obvious because the image is the boundary of the four-chain Z^a. Secondly, a D3-brane wrapping on ∂Z^a has angular momentum δ_b^a with respect to the isometry v_b. It is because

$$\int_{\partial Z^a} \omega_b = \int_{Z^a} d\omega_b = \int_{Z^a} \iota_{k_b} \operatorname{vol}^\circ = \delta_b^a. \tag{8.8.7}$$

For the sake of completeness, we would like to describe the second arrow ι in (8.8.2) and in (8.8.3) in the toric case. Let us denote the cone over X by $C(X)$, which is a toric variety. For $u \in M$, we can take a rational function χ^u on $C(X)$ satisfying

$$v^i \partial_i \chi^u = \sqrt{-1} \langle u, v \rangle \chi^u \tag{8.8.8}$$

for $v \in N$, where $\langle u, v \rangle$ is the natural pairing between M and N. It is unique up to multiplication by a complex number, since the torus action is dense in $C(X)$. Then the image is precisely the principal divisor $\operatorname{div}(\chi^u)$ determined by χ^u restricted on X, where the principal divisor $\operatorname{div}(f)$ of a rational function f is

$$\operatorname{div}(f) = \sum_\alpha n_\alpha C^\alpha, \tag{8.8.9}$$

with C^α the loci of the zeros and the poles of f and with n_α the degree of zeros or the negative of the degree of poles at n_α.

Chapter 9

Conclusion

Recent developments made it apparent that at the basis of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence with eight supercharges lies the extremization principle, which determines the superconformal R-symmetry. Uncovering its details was the main theme of the thesis.

The correspondence can be viewed from three perspectives, namely from that of the 4d SCFT, from that of the 5d gauged supergravity, and from the geometry of the five-dimensional Sasaki-Einstein manifold. The extremization principle appears in each of the perspectives in different guises:

- In the 4d SCFT it is the a-maximization reviewed in section 3.4. The basic problem is the identification of the correct linear combination of the global symmetries to be the R-symmetry, which was done through the maximization of the trial central charge.

- In the 5d gauged supergravity, the vacuum expectation values of the vector multiplet scalars were found by the minimization of the superpotential. They in turn determine the linear combination of the gauge field which minimally couples to the gravitino. The procedure was explained in detail in section 5.2.

- In the 5d Sasaki-Einstein geometry, the R-symmetry corresponds to the Reeb vector which is an isometry canonically defined by the metric. Although the determination of the metric involves the solution to the highly non-linear Monge-Ampère equation, the Reeb vector can be found through the Z minimization, as reviewed in section 6.5.

We saw in section 5.2 that the a-maximization in 4d SCFT and the superpotential minimization in 5d gauged supergravity correspond naturally under the GKP-W prescription. To connect the a-maximization to the Z-minimization, one first needs to study how the quiver gauge theory is determined by the toric data of the Calabi-Yau cone, and how the triangle anomaly of the quiver theory is related to the geometry of the Sasaki-Einstein manifold. The solution to the first problem is reviewed in chapter 7, while the second was fully explored in chapter 8. Then the relation of the a-maximization to the Z-minimization was explained in section 8.6 for toric Sasaki-Einstein manifolds. The situation is schematically depicted in figure 9.1.

There are several unsatisfactory points in the understanding. Firstly, the proof of the equivalence of the a-maximization and the Z-minimization was done by a brute force calculation. It should be able to deduce the relation in a more physical way. Indeed, since we have a physical understanding of the relation between the 4d a-maximization and the 5d P-minimization, what needs to be done is to connect the P-minimization to the Z-minimization. Indeed, the Z-minimization is the minimization of the volume in the space of Sasaki metrics. If on-shell, the volume of the Sasaki-Einstein manifold

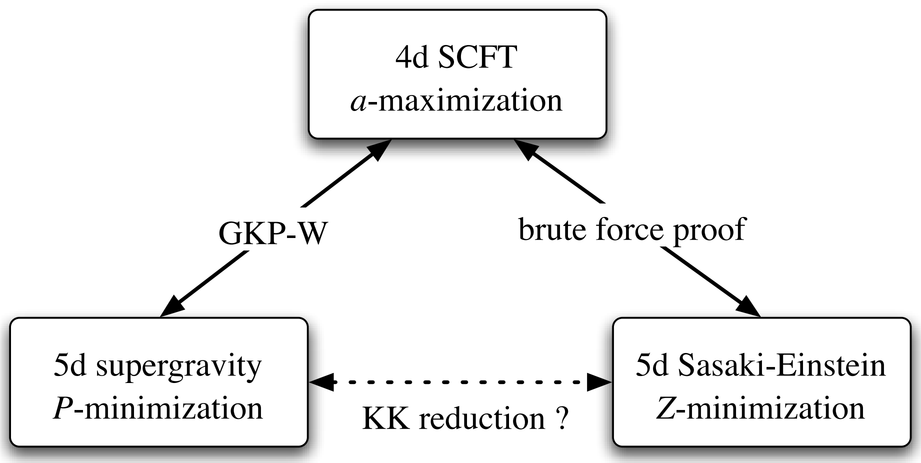

Figure 9.1: The status of the three extremization principle.

is directly related, through the Kaluza-Klein reduction, to the cosmological constant of the 5d theory on the AdS space, which equals to $-P^2$. The result in section 8.6 implies the correspondence holds even for slightly off-shell metrics. It is natural to suspect that the relation can be established by doing an off-shell analysis of the Kaluza-Klein reduction. The reduction for the gauge fields was done and explained in chapter 8. Thus, we need to extend the analysis to include the scalar fields. It would be difficult, but should be doable if we utilize the very special structure of the scalar manifold.

Another point to be discussed is the higher derivative corrections in the 5d gauged supergravity. In relating the a-maximization to the P-minimization, we assumed that the difference $a - c$ of the two central charges is small, since it corresponds to the interaction $\int A \wedge \operatorname{tr} R \wedge R$. It is a higher derivative effect in 5d supergravity, while a $1/N$ or $1/N^2$ effect in the 4d SCFT. It was already included in the a-maximization as the $\operatorname{tr} R(s)$ term. Unfortunately the 5d gauged supergravity structure of the corresponding term has not been studied in the literature. It should be possible, however, to analyze the correction in the 5d bulk, because it also is a protected term in the Lagrangian through the consideration of the anomaly. We hope to revisit these problems in the future.

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