Black Hole Entropy in the presence of Chern-Simons terms

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to appear in Class. Quant. Grav.
Contents

1. Introduction

2. Black Hole Spacetime

3. Black Hole Spacetime in Higher Derivative Gravity

4. Derivation of the First law

5. Summary
• Classical General Relativity leads to
\[ \frac{\kappa}{2\pi} \frac{\delta A}{4G_N} = \delta m - \Omega \delta J \]

• Semiclassical analysis identifies
\[ T_H = \frac{\kappa}{2\pi} \]

• Very natural to identify
\[ S = \frac{A}{4G_N} \]

and look for statistical explanation.
• Extremal charged black hole $\sim$ D-branes

• Excitations can be counted, account for

$$S = \frac{A}{4G_N}$$

[Strominger-Vafa], [Maldacena-Strominger-Witten]
• Extremal charged black hole $\sim$ D-branes

• Excitations can be counted, account for

$$S = \frac{A}{4G_N} + \cdots$$

[Strominger-Vafa], [Maldacena-Strominger-Witten]

• Also predicts subleading corrections.
Corrections to the area law

- Einstein-Hilbert
  \[ \mathcal{L} = \sqrt{-g} \frac{R}{16\pi G_N} \]

- Area law
  \[ S = \frac{A}{4G_N} \]
Corrections to the area law

- Einstein-Hilbert corrected:

\[ \mathcal{L} = \sqrt{-g} \left( \frac{R}{16\pi G_N} + \frac{c}{2} R^2 + \cdots \right) \]

- Area law accordingly modified:

\[ S = \frac{A}{4G_N} + 8\pi c \int_{\text{hor}} R_{rtrt} + \cdots \]
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S = \frac{A}{4G_N} + 8\pi c \int_{\text{hor}} R_{rtrt} + \cdots
\]

- Wald’s formula:

\[
S = -2\pi \int_{\text{hor}} \sqrt{-g} \frac{\delta \mathcal{L}}{\delta R_{abcd}} \epsilon_{ab} \epsilon_{cd}
\]
String theory works

- Two ways to calculate corrections:
  - Microscopic: d.o.f. on the brane
  - Macroscopic: Wald’s formula

- Completely agrees!

[de Wit et al.][Ooguri-Strominger-Vafa]
String theory works

- Two way to calculate corrections:
  - Microscopic
    - d.o.f. on the brane
  - Macroscopic
    - Wald’s formula

- Completely agrees!

[de Wit et al.][Ooguri-Strominger-Vafa]

- In four dimensions.
Odd dimensions

- Black rings in 5d.
- Entropy correction:

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- Why?
Odd dimensions

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- Why? Wald’s formula isn’t applicable to gravitational Chern-Simons:

\[
\int F \wedge \text{tr } \Gamma \wedge R
\]
Odd dimensions

- Black rings in 5d.
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Our aim today

Entropy correction from grav. CS.
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5. Summary
• Killing Horizon: $\xi^2 = 0$.

• Binormal

$$\epsilon_{ab} = \xi_a n_b - \xi_b n_a$$

• Surface gravity

$$\nabla_a \xi_b = \kappa \epsilon_{ab} + \cdots$$
• **Bifurcation Surface**
  \[ \xi = 0. \]

• **Bifurcate horizon**: A pair of horizons which pass BS.
Hawking radiation

- Free field radiates at
  \[ T_H = \frac{\kappa}{2\pi}. \]
- Unruh effect near B.S.
- Depends only on bg metric,
- Not quantum gravity per se.

\[ \xi = 0 \]
\[ \xi^2 = 0 \]

Yuji Tachikawa (SNS, IAS)
• Horizon cross section at finite $t$ is the b.s.
Holonomy at B.S.

- At the B.S.,

\[ \nabla_c \epsilon_{ab} = \kappa^{-1} \nabla_c \nabla_a \xi_b = \kappa^{-1} R_{abcd} \xi^d = 0. \]

\[ \text{holonomy reduces to} \]

\[ SO(D - 1, 1) \subset SO(1, 1)_N \times SO(D - 2) \]
Black Hole Thermodynamics

Zeroth law
\[ \kappa \text{ constant on the horizon.} \]

First law
\[ \frac{\kappa}{2\pi} \frac{\delta A}{4G_N} = \delta m - \Omega \delta J \]

Second law
\[ A \text{ increases with time.} \]
3. Black Hole Spacetime in Higher Derivative Gravity
Higher derivative corrections

- **RG perspective**
  - Planck suppressed terms $R^2$, $R^4$ etc.

- Coefficients calculable in string compactifications

- Affects black hole solutions.
Conceptual problems

- Null dir. of $g_{ab} \neq$ maximal propagation speed.
- e.g. Field redefinition

$$g_{ab} \rightarrow g_{ab} + cR_{ab} + c'R_{g_{ab}} + \cdots$$

changes the ‘light cone’

- Physics should be invariant!
• Define the horizon using the light cone of $g_{ab}$.

• Suppose it has Killing, bifurcate horizon:
  
  under $g_{ab} \rightarrow g_{ab} + h_{ab}$,

  Bifurcation surface
  
  invariant, defined by $\xi^{a} = 0$

  Bifurcate horizon
  
  $\xi^{a}\xi_{a} \rightarrow g_{ab}\xi^{a}\xi^{b} + h_{ab}\xi^{a}\xi^{b}$.

  2nd term invariant under $\xi$

  $\rightarrow$ zero by evaluating at B.S.

Hawking temperature

Evaluate $\kappa^{2} = |\nabla_{a}\xi^{b}|^{2}$ at B.S.

$\rightarrow$ Christoffel drops off because $\xi^{a} = 0$. 
Black Hole Thermo. in Higher Derivative Gravity

Zeroth law
\( \kappa \) constant on the horizon: assumption

First law
\[ \frac{\kappa}{2\pi} S = \delta m - \Omega \delta J, \quad S \text{ given by } \text{Wald's formula} \]

Second law
Difficult to establish
Wald’s formula

Black Hole Entropy

\[ S = -2\pi \int_{\text{hor}} \sqrt{-g} \frac{\delta \mathcal{L}}{\delta R_{abcd}} \epsilon_{ab} \epsilon_{cd} \]

for \( \mathcal{L} = \mathcal{L}(g_{ab}, R_{abcd}, \nabla_e R_{abcd}, \cdots ; \phi, \cdots) \)

- It does not include gravitational Chern-Simons
  \[ \ast \delta \mathcal{L} = \text{tr} \, \Gamma \wedge R^{2n-1} \]

- or Green-Schwarz type coupling
  \[ \delta \mathcal{L} = \frac{1}{2} |H|^2 \quad \text{with} \quad H = dB + \text{tr} \, \Gamma \wedge R. \]
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**Notation**

\( D \)  space-time dimension

\( L(\phi) \)  Lagrangian density as \( D \)-form

\( \phi \)  collective symbol for the fields. \( g_{\mu\nu}, A_\mu, \ldots \)

\( \mathcal{L}_\xi \)  Lie derivative by a vector field \( \xi \),

\[
\mathcal{L}_\xi \omega = (d\iota_\xi + \iota_\xi d)\omega.
\]

\( \delta_\xi \)  Variation under diffeo. e.g.

\[
\delta_\xi \Gamma^a_b = \mathcal{L}_\xi \Gamma^a_b + d(U_\xi)^a_b
\]

where \((U_\xi)^a_b = \partial_b \xi^a\).
• Assumption by Wald:

$$\delta_\xi L(\phi) = \mathcal{L}_\xi L(\phi).$$

Cannot incorporate the CS.

• Today:

$$\delta_\xi L(\phi) = \mathcal{L}_\xi L(\phi) + d\Xi_\xi.$$

• Almost verbatim transcript of Wald’s original.
Covariant Hamiltonian Method

- EOM $E_\phi$ and symplectic potential $\Theta$ via

$$\delta L = E_\phi \delta \phi + d\Theta(\phi, \delta \phi)$$

- $\Theta$ pairs of ‘coordinate’ and ‘momenta’

$$L(\phi) = \frac{1}{2} \ast d\phi \wedge d\phi \quad \Theta = \ast d\phi \wedge \delta \phi$$

- Symplectic form given by

$$\Omega(\phi, \delta_1 \phi, \delta_2 \phi) = \delta_1 \Theta(\phi, \delta_2 \phi) - \delta_2 \Theta(\phi, \delta_1 \phi).$$
Lemma

- \( j \): a form constructed from
  - fields \( \phi \)
  - an external field \( \xi \)
- suppose \( j \) is closed on-shell for any \( \xi \).
- It is then exact on-shell. i.e.

\[
dj_\xi \simeq 0 \rightarrow j_\xi \simeq dQ_\xi.
\]

- \( \simeq \): equality on-shell.
Noether’s theorem

• Symmetry leads to conserved current:

\[ j_\xi = \Theta(\phi, \delta_\xi \phi) - \iota_\xi L - \Xi_\xi \]

satisfies

\[ dj_\xi \simeq 0 \]

• The lemma implies

\[ j_\xi \simeq dQ_\xi, \quad \int_C j_\xi \simeq \int_{\partial C} Q_\xi. \]
Noether’s theorem

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i.e.

Global symmetry \quad \rightarrow \quad Conserved charges
Gauge symmetry \quad \rightarrow \quad Gauss law
Un-illuminating main part

- Define $\Pi_\xi$ via
  \[ \delta_\xi \Theta = \mathcal{L}_\xi \Theta + \Pi_\xi. \]
- Recall $\delta L = E_\phi \delta \phi + d\Theta$, $\delta_\xi = \mathcal{L}_\xi L + d\Xi_\xi$. 
• Define $\Pi_\xi$ via

$$\delta_\xi \Theta = \mathcal{L}_\xi \Theta + \Pi_\xi.$$ 

• Recall $\delta L = E_\phi \delta \phi + d\Theta, \quad \delta_\xi = \mathcal{L}_\xi L + d\Xi_\xi$.

• Calculating $\delta \delta_\xi L$ in two ways:

$$d\Pi_\xi \simeq \delta d \Xi_\xi \quad \rightarrow \quad \Pi_\xi - \delta \Xi_\xi \simeq d\Sigma_\xi.$$ 

$$\delta j_\xi = \delta \Theta(\phi, \delta_\xi \phi) - \iota_\xi \delta L - \delta \Xi_\xi$$

$$\simeq \delta \Theta(\phi, \delta_\xi \phi) - \delta_\xi \Theta(\phi, \delta \phi) + d\iota_\xi \Theta + \Pi_\xi - \delta \Xi_\xi$$

$$\simeq \Omega(\phi, \delta \phi, \delta_\xi \phi) + d(\iota_\xi \Theta + \Sigma_\xi).$$
• Define $\Pi_\xi$ via
\[
\delta_\xi \Theta = \mathcal{L}_\xi \Theta + \Pi_\xi.
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• Recall $\delta L = E_\phi \delta \phi + d\Theta$, $\delta_\xi = \mathcal{L}_\xi L + d\Xi_\xi$.

• Calculating $\delta \delta_\xi L$ in two ways:
\[
d\Pi_\xi \simeq \delta d \Xi_\xi \longrightarrow \Pi_\xi - \delta \Xi_\xi \simeq d\Sigma_\xi.
\]
\[
\delta j_\xi = \delta \Theta(\phi, \delta_\xi \phi) - \nu_\xi \delta L - \delta \Xi_\xi
\]
\[
\simeq \delta \Theta(\phi, \delta_\xi \phi) - \delta_\xi \Theta(\phi, \delta \phi) + d\nu_\xi \Theta + \Pi_\xi - \delta \Xi_\xi
\]
\[
\simeq \Omega(\phi, \delta \phi, \delta_\xi \phi) + d(\nu_\xi \Theta + \Sigma_\xi).
\]

• Thus, for $C_\xi$ with $\delta C_\xi = \nu_\xi \Theta + \Sigma_\xi$,
\[
\delta dQ'_\xi \simeq \Omega(\phi, \delta \phi, \delta_\xi \phi)
\]
where
\[
Q'_\xi = Q_\xi - C_\xi.
\]
Recap.

- $\delta dQ'_\xi \simeq \Omega(\phi, \delta \phi, \delta \xi \phi)$ means
  \[
  \int dQ'_\xi \text{ is the Hamiltonian generating } \xi.
  \]

- Let $\xi = t + \Omega \phi$ where
  - $\xi$ Horizon generating Killing
  - $t$ global time translation
  - $\phi$ angular rotation

Then

\[
\delta \int_{\text{hor}} Q'_\xi \simeq \delta \int_{\infty} Q'_t + \Omega \delta \int_{\infty} Q'_\phi.
\]
First Law

\[ \delta \int_{\text{hor}} Q'_\xi \simeq \delta \int_{\infty} Q'_t + \Omega \delta \int_{\infty} Q'_\phi. \]

- \( E = \int_{\infty} Q'_t, \ J = \delta \int_{\infty} Q'_\phi. \)
- Taking the horizon at the bifurcation surface,

\[ \int_{\text{hor}} Q'_\xi = \kappa \int_{\text{hor}} Q'_\xi \bigg|_{\xi \to 0, \nabla_a \xi_b \to \epsilon_{ab}} \]

- Then, define

\[ S = 2\pi \int_{\text{hor}} Q'_\xi \bigg|_{\xi \to 0, \nabla_a \xi_b \to \epsilon_{ab}} \]

Result.

\[ \frac{\kappa}{2\pi} \delta S = \delta E + \Omega \delta J. \]
3d Chern-Simons

- Start from $L_{CS} = \beta \text{tr}(\Gamma R - \frac{1}{3} \Gamma^3)$

  $\delta_\xi L_{CS} = \mathcal{L}_\xi L_{CS} - d(\beta \text{tr} dU_\xi \Gamma) \quad \Xi_\xi = -\beta \text{tr} dU_\xi \Gamma$

- Non-covariant part in $\Theta$ is $-\beta \text{tr} \Gamma \delta \Gamma$

  $\Pi_\xi = -\beta \text{tr} dU_\xi \delta \Gamma$

  $\Pi_\xi - \delta \Xi_\xi \simeq 0 \quad \Sigma_\xi = 0$

  $j_\xi = \Theta(\phi, \delta_\xi \phi) - \Xi_\xi + \cdots = 2\beta \text{tr} dU_\xi \Gamma + \cdots$

  $Q'_\xi = 2\beta \text{tr} U_\xi \Gamma + \cdots$

- $(U_\xi)^a_b = \partial_b \xi^a$. 
3d Chern-Simons

- \( L_{CS} = \beta \text{tr}(\Gamma R - \frac{1}{3} \Gamma^3) \) \( \Rightarrow \) \( S_{CS} = 8\pi \beta \int_{\text{hor}} \Gamma_N \)

where \( \Gamma_N = -\epsilon^{\nu \mu} \Gamma_{\nu \rho} dx^\rho / 2 \).

- agrees with known corrections to BTZ.

- Naive application of Wald \( \Rightarrow 4\pi \beta \int_{\text{hor}} \Gamma_N \).
Chern-Simons in General Dimensions

- \( L_{CS} = \beta \ tr(\Gamma R^{2m-1} + \cdots) \)

\[ S_{CS} = 8\pi m\beta \int_{\text{hor}} \Gamma_N R_N^{2m-2} \]

- Recall

\text{Holonomy at the bifurcation surface} \quad = \quad SO(1,1)_N \times SO(D - 2)

- Entropy correction = CS of the normal bundle!
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Summary

- Spacetime with black holes reviewed.
- Entropy correction from the grav. CS.
Summary

• Spacetime with black holes reviewed.
• Entropy correction from the grav. CS.

Outlook – Need application!

• Black rings. – working on it with friends.
• 7d black holes. – idea wanted.

\[
\text{tr } \Gamma \wedge R^3 \longrightarrow \int_{\text{hor}} \Gamma_N \wedge R^2_N
\]