On asymptotic density of states in 2d CFT with symmetry

Yuji Tachikawa (IPMU)

roughly based on 2208.05495 in collaboration with

Ying-Hsuan Lin, Masaki Okada, Sahand Seifnashri

It is a great honor to speak in this special occasion.

I am a string theorist, and am neither a collaborator nor a former student.

Not many of you know me, I am afraid.

But at least I've known Yasu/河東先生 for a quarter century.

My first in-person encounter with Kawahigashi-sensei is in the spring of 1998, when I became an undergraduate at U. Tokyo.

Back then, he was several years younger than I am now.

Time really flies.

In the first semester of my university life in 1998,
I took his special course in calculus using non-standard analysis.

That was a revelation to me. It was a baptism by pure math.

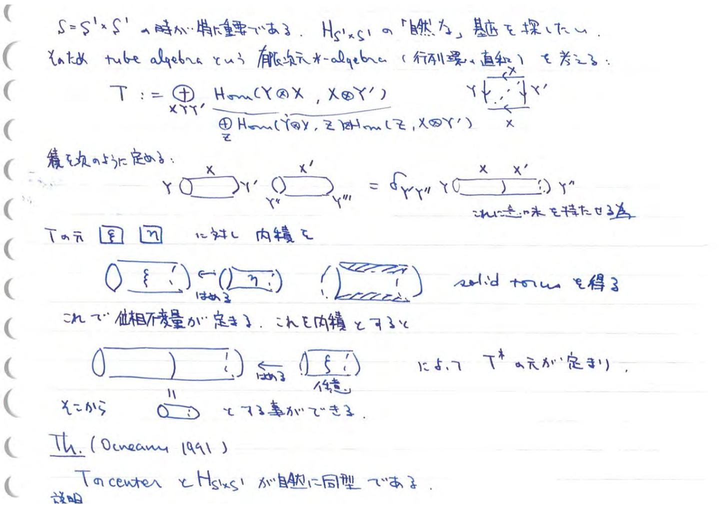
Unfortunately I couldn't find the notes I took on this course.

Instead I found the notes from a different course of his, given in the next year in 1999, on his then-recent works.

The first few lectures were about the Turaev-Viro TQFTs associated to an arbitrary unitary fusion category
(if I use a historically inaccurate but now-common terminology).

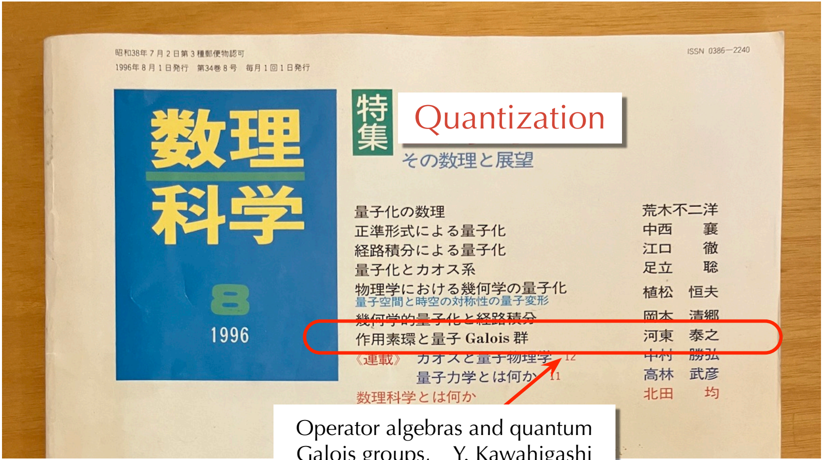
The latter part was about α-induction.

This is a page from the notes, on the **tube algebra**:



My virtual encounter with Kawahigashi-san goes back a few more years, when I read his article in the magazine「数理科学」, August issue, 1996.

There he wrote passionately about paragroups
and the theory of subfactors.



Operator algebras and quantum Galois groups, Y. Kawahigashi

(It is also available at `https://www.ms.u-tokyo.ac.jp/~yasuyuki/suri9608.pdf`)

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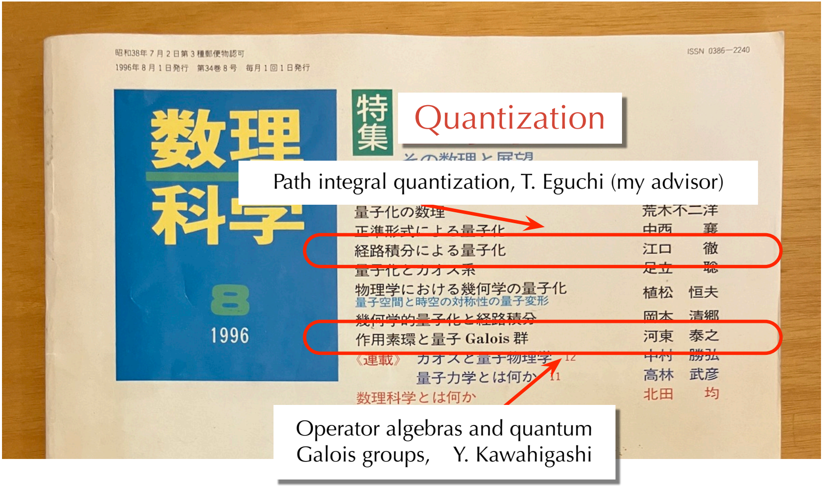
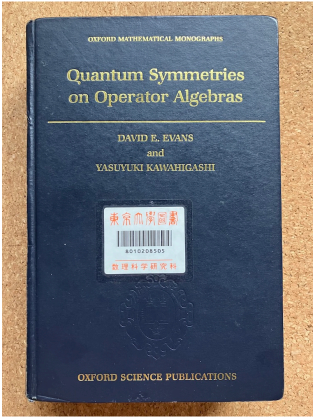


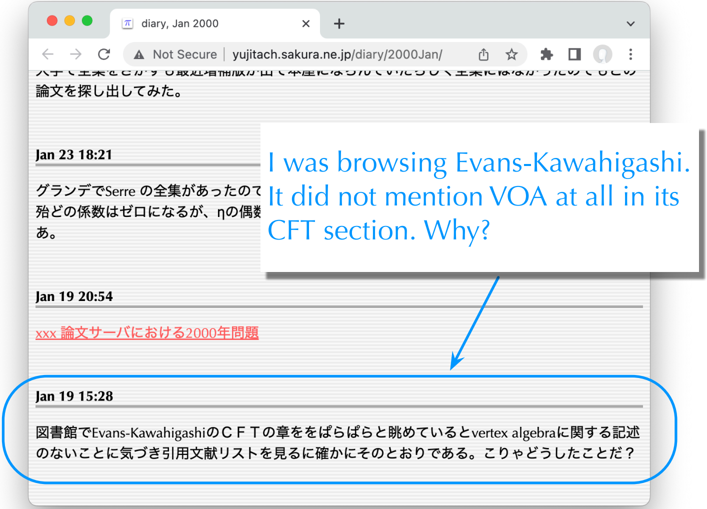
(It is also available at `https://www.ms.u-tokyo.ac.jp/~yasuyuki/suri9608.pdf`)

Eventually, I decided to pursue theoretical physics rather than pure math, but Kawahigashi-sensei's works have remained an inspiration to me.

E.g. I have tried to read Evans-Kawahigashi ...



My journal entry from January 19, 2000:



I got a reply from Kawahigashi-sensei a few days afterwards!

From: **Yasuyuki Kawahigashi** yasuyuki@ms.u-tokyo.ac.jp
Subject:
Date: January 27, 2000 11:41
To: g841061@mail.ecc.u-tokyo.ac.jp, yasuyuki@ms.u-tokyo.ac.jp

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立川君,
そちらのホームページ(前から知ってました)にぼくの本のことが書いてある
のに気づいたので，疑問に答えましょう．場の量子論に対する作用素環的ア
プローチと，vertex (operator) algebra の間に何らかの関係があるであろ
うというのは誰でもそう思うんですが，我々があの本を書いている段階では
具体的な関係は一つも知られていませんでした．現在でも，一応具体的な関
係と言えるのは，Feng Xu による次の2本の論文しかないと思います．まだ
まだほとんど何もわかっていないというのが現状です．
Algebraic Coset Conformal Field Theories I, math.OA/9810035.
Algebraic Coset Conformal Field Theories II, math.OA/9903096.
一方，CFT と vertex (operator) algebra の間にはもちろん関係があるわ
けですが，作用素環と直接関係ないことはあの本のテーマでもないし，また
そもそも我々には(あるいは少なくともぼくには)まともなことは書けないと
いうことで，あの本には何も書いてないのです．まあ，それでも参考文献表
に何か載せておいた方がよかったですね．
河東泰之

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Subject:
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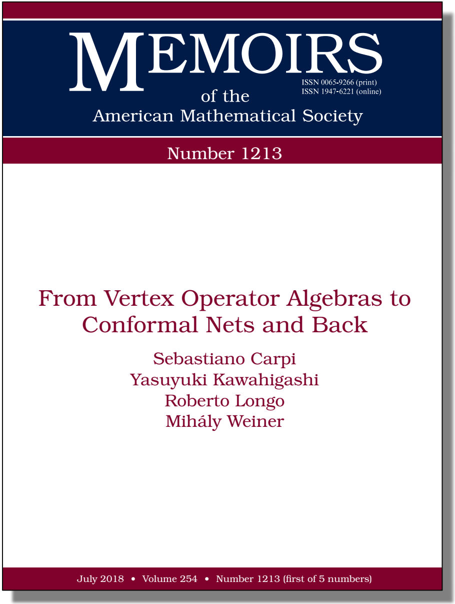
立川君,

Yes everyone thinks that there should be relations between VOA and AQFT, but no relations were known when the book was written. Even now, the only concrete results are the following two papers by Feng Xu …

に何か載せておいた方がよかったですね.
河東泰之

Of course you know the seminal work appeared in 2018 ...



All these years, I have been working on QFTs and on string theory from non-rigorous, theoretical-physics points of view instead.

As an outsider who has always looked upon him from afar, it really is a special honor for me to speak at this wonderful event this week.

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Happy return of the calendar (還暦), Kawahigashi-sensei!

For non-East Asians in the audience: in Chinese calendar,
there are **10-year cycle** (十干)

甲	乙	丙	丁	戊	己	庚	辛	壬	癸
🪵↑	🪵↓	🔥↑	🔥↓	🧱↑	🧱↓	💰↑	💰↓	💦↑	💦↓

and **12-year cycle** (十二支)

子	丑	寅	卯	辰	巳	午	未	申	酉	戌	亥
🐭	🐮	🐯	🐰	🐲	🐍	🐴	🐑	🐵	🐔	🐶	🐗

going as

$$\text{甲子}, \text{乙丑}, \cdots \text{癸亥}$$

coming back every $\mathbf{gcd(10, 12) = 60}$ years.

This is why the 60-year anniversary is called
還暦 = the return of the calendar in Japan.

(Yasu was born in 1962, which was 壬寅. So 2022 is also 壬寅.
This year 2023 is 癸卯.)

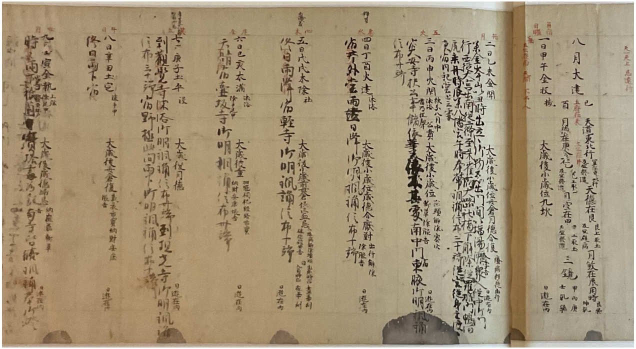
Speaking of 10-year and 12-year cycles...

Did you know that the **7-day cycle** originated in Babylonia and was brought west to Europe and east to China/Japan independently?

Sunday	日曜日	(day of the sun)
Monday	月曜日	(day of the moon)
Tuesday	火曜日	(day of the planet Mars)
⋮	⋮	⋮
Saturday	土曜日	(day of the planet Saturn)

Tues is an Anglo-Saxon god equivalent to Mars, etc.

The 7-day cycle was in use already in Japan,
long before our contact with Europeans:



And the days of the week of East Asians and of the Europeans have been **in sync**!

(This is a page from the diary of the regent 藤原道長 by his own hand,
the entry of 寛弘四年八月二日 = 1007.Sep.15 in Julian calendar. It's 371028 days before today, and is a Monday= 月曜日.

Taken from [近衛家名宝からたどる宮廷文化史 (笠間書房)pp.ii-iii]. Also available at `https://dl.ndl.go.jp/pid/2591253/1/11`)

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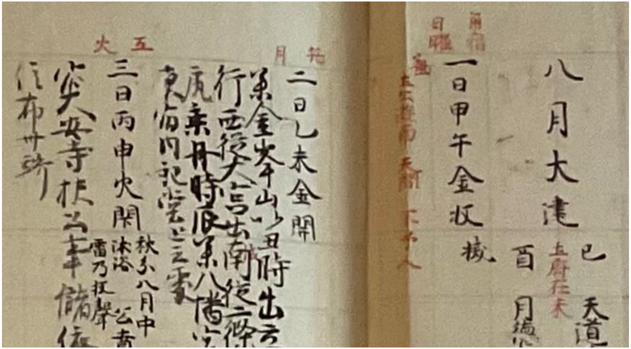

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Hmm, what was I supposed to talk about?

Today I'd like to talk about certain general properties of QFTs conjectured by physicists in a manner hopefully understandable to practitioners of AQFT.

It tries to connect **asymptotic density of states** of a QFT and **quantum dimension of sectors** of a net of algebras.

Consider a d-dim'l QFT with G symmetry.

Put it on $M_{d-1} \times \mathbb{R}_{\text{time}}$, **where M_{d-1} is compact**.

The Hilbert space $\mathcal{H}$ of the system can be decomposed as

$$\mathcal{H} = \bigoplus_{\text{irrep } \rho} \mathcal{H}_\rho.$$

Question: What is the "relative size" of $\mathcal{H}_\rho$ w.r.t. the whole of $\mathcal{H}$?

$\dim \mathcal{H}_\rho$ is infinite, so $\dim \mathcal{H}_\rho / \dim \mathcal{H}$ does not make sense.

Let us instead consider

$$\mathrm{tr}_{\mathcal{H}_\rho} e^{-\beta H}$$

where H is the Hamiltonian, the generator of the time translation.

(As the spatial slice M_{d-1} is assumed to be compact, the spectrum of H would be discrete, and so this expression should make sense.)

We then consider the ratio

$$\text{“}\frac{\dim \mathcal{H}_\rho}{\dim \mathcal{H}}\text{”} := \lim_{\beta \to 0} \frac{\mathrm{tr}_{\mathcal{H}_\rho} e^{-\beta H}}{\mathrm{tr}_{\mathcal{H}} e^{-\beta H}}$$

This is the **asymptotic density of states** in the title.

Conjecture

Take a d-dim'l QFT with G symmetry on $M_{d-1} \times \mathbb{R}_{\text{time}}$
where M_{d-1} is compact.

Assume G is finite and acts faithfully on $\mathcal{H}$.
Decompose $\mathcal{H} = \bigoplus_{\text{irrep } \rho} \mathcal{H}_\rho$. Then we universally have

$$\lim_{\beta \to 0} \frac{\operatorname{tr}_{\mathcal{H}_\rho} e^{-\beta H}}{\operatorname{tr}_{\mathcal{H}} e^{-\beta H}} = \frac{(\dim \rho)^2}{|G|}$$

independent of QFT.

[Pal-Sun 2004.12557]	Physics derivation in 2d
[Harlow-Ooguri 2109.03838]	Conjecture in general dimension
[Magan 2111.02418]	Purported proof in general dimension which I do not understand

Question to AQFT persons

Can it be proved this conjecture with AQFT methods?

Caveat:

AQFT is usually done on noncompact spatial slice, on $\mathbb{R}^{d-1} \times \mathbb{R}_{\text{time}}$.

There have been some works on 2d models on $S^1 \times \mathbb{R}_{\text{time}}$, though.

So let me concentrate on this case today.

Another caveat:

Due to long years of independent developments,

$$\begin{aligned} &\text{QFT terminologies among hep-th persons} \\ \neq\ &\text{QFT terminologies among AQFT persons} \end{aligned}$$

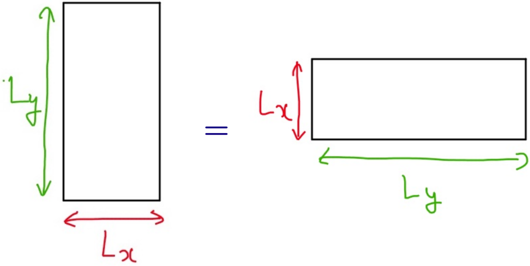
So some more translations are necessary.

e.g. **Flavor symmetry (hep-th)** = **Global gauge symmetry (AQFT)**

From this page and onwards, I try to use the AQFT language,
without actually understanding it.

Please correct me when I say something wrong.

2d QFT for hep-th people (and for mathematicians adopting Atiyah-Segal-type bordism approach) **is supposed to be modular invariant on Euclidean T^2:**



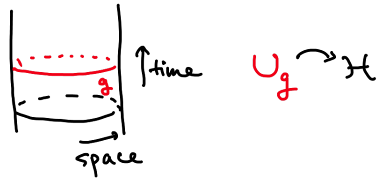
$$\mathrm{tr}_{\mathcal{H}_{L_x}} e^{-L_y H_{L_x}} = \mathrm{tr}_{\mathcal{H}_{L_y}} e^{-L_x H_{L_y}}$$

This corresponds, in the **chiral** conformal case,
(i.e. containing only $\mathbf{Vir}$),
to a **holomorphic conformal net $\mathcal{A}$**.

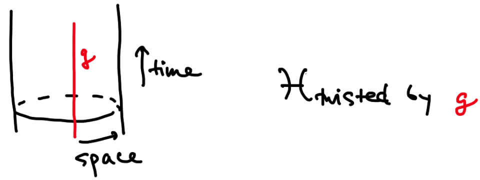
In the **non-chiral** conformal case (i.e. containing $\mathbf{Vir} \otimes \mathbf{Vir}$),
it would correspond to a **net $\mathcal{A}$ with trivial $\mathrm{DHR}(\mathcal{A})$**.

(In other words, what hep-th people consider as the algebra of operators of a QFT is the maximally extended one.)

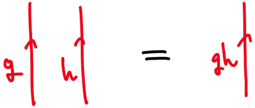
In 2d, a group action can be represented by a horizontal line:



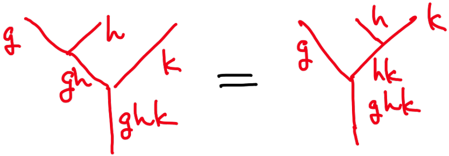
Twisted boundary conditions are represented by vertical lines:



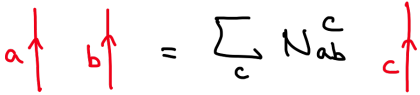
These topological lines can be fused:



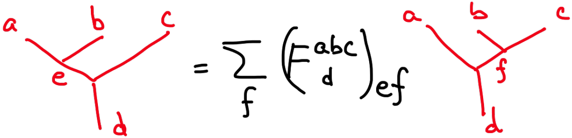
and this fusion is associative:



This can be generalized so that the fusion can be a sum



then the associativity is given by



which needs to satisfy the pentagon equation.

A package of topological walls with these data
gives a **generalized version of finite group symmetry** in 2d.

Mathematically given by a **unitary fusion category**.

(The concept was already heavily used by ca. 1995
by Evans and Kawahigashi; this terminology is more recent ca. 2005
and is due to Etingof-Nykshych-Ostrik.)

In hep-th such symmetries are now called **non-invertible symmetries**.

In AQFT literature, they were formulated in terms of **hypergroup actions** by Bischoff and collaborators.

Assuming $\mathbf{DHR}(\mathcal{A})$ is trivial, a **hypergroup action** on $\mathcal{A}$ is the **action of the fusion algebra** of a fusion category $\mathbf{F}$.

The fusion category $\mathbf{F}$ also describes **transparent boundary conditions** between $\mathcal{A}$ and $\mathcal{A}$.

Furthermore, the fixed point subnet $\mathcal{B} = \mathcal{A}^{\mathbf{F}}$ is such that

$$\mathbf{DHR}(\mathcal{B}) = Z(\mathbf{F}),$$

where $Z(\mathbf{F})$ is the Drinfeld center of $\mathbf{F}$.

[Bischoff-Kawahigashi-Longo-Rehren, 1405.7863]
[Bischoff, 1608.00253]
[Bischoff-Rehren, 1612.02972]

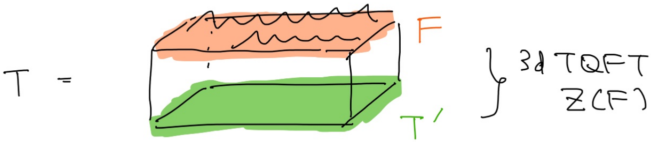
Hep-th people in the last couple of years say:

$\mathcal{A}$ (with trivial $\mathbf{DHR}(\mathcal{A})$) is the algebra of point operators of an (absolute) QFT T with $\mathbf{F}$ symmetry

and

$\mathcal{B} = \mathcal{A}^{\mathbf{F}}$ is the algebra of point operators of a **relative** QFT T' whose symmetry TFT is $Z(\mathbf{F})$

Dan Freed likes to call T as a quiche:



see e.g. [Freed, 2212.00195]

Study of symmetries more general than groups has a long history in AQFT.

Study of such generalized symmetries became active in hep-th/cond-mat only in the last couple of years.

e.g. there are some numerical works constructing **2d non-chiral CFTs** having the Haagerup fusion category as symmetry. ($c_L = c_R \sim 2.0$)

[Huang-Lin-Ohmori-YT-Tezuka, 2110.03008]
[Vanhove-Lootens-Van Damme-Wolf-Osborne-Haegeman-Verstraete, 2110.03532]

c.f. Evans and Gannon have been long looking for **2d chiral CFTs** having the Haagerup symmetry. ($c_L = 8k$, $c_R = 0$)

People have also found that **such non-invertible symmetries are abound in general dimensions, e.g. in 4d massless quantum electrodynamics.**

[Choi-Córdova-Hsin-Lam-Shao 2111.01139]
[Kaidi-Ohmori-Zheng 2111.01141]

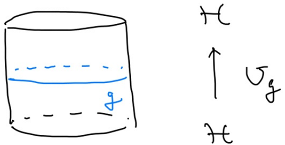
This might sound contradictory to the classic AQFT results of Doplicher-Haag-Roberts or Buchholz-Fredenhagen, who found that a symmetry structure is always an ordinary group in dimension 4.

But the type of charges hep-th people are considering can only be localized on a half-space, so there is no immediate contradiction.

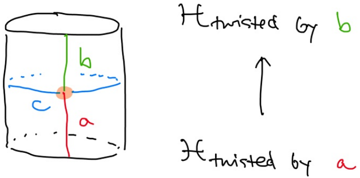
This means that studies of less-localizable charges in higher dimensions in AQFT might be of some interest ...

Anyway, let's finish this digression and come back to our conjecture!

How does $\mathbf{F}$ act on states? The action of a group element

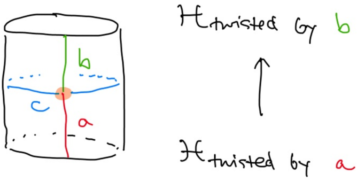


is generalized for a fusion category symmetry $\mathbf{F}$ to an action of



where $\bullet \in \mathbf{Hom}(a \otimes c, c \otimes b)$.

The operators

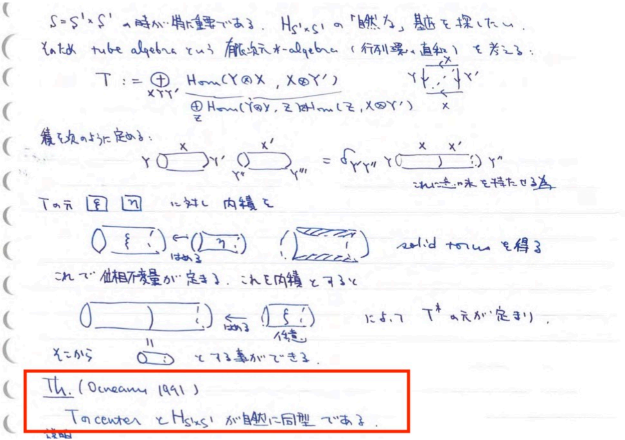


for ● $\in \mathbf{Hom}(\boldsymbol{a} \otimes \boldsymbol{c}, \boldsymbol{c} \otimes \boldsymbol{b})$
generate the tube algebra $\mathbf{Tube}(\mathbf{F})$ of the fusion category $\mathbf{F}$,
and therefore

$$\bigoplus_{\boldsymbol{a}:\ \text{simples of } \mathbf{F}} \mathcal{H}_{\text{twisted by } \boldsymbol{a}}$$

are decomposed in terms of **irreps of** $\mathbf{Tube}(\mathbf{F})$,
which are 1-to-1 with the **simples of** $Z(\mathbf{F})$.

Note that this relation between **irreps of Tube(F)** and **simples of Z(F)** is exactly what I learned from Yasu in 1999:


$S = S^1 \times S^1$ の時が特に重要である．$H_{S^1 \times S^1}$ の「自然な」基底を探したい．

そのため tube algebra という有限次元 $*$-algebra（行列環の直和）を考える：

$$T := \bigoplus_{X,Y,Y'} \mathrm{Hom}(Y \otimes X, X \otimes Y') = \bigoplus_Z \mathrm{Hom}(Y \otimes X, Z) \otimes \mathrm{Hom}(Z, X \otimes Y')$$

積を次のように定める：

$= \delta_{Y'Y''}$

これに意味を持たせる為

Tの元 ξ η に対し 内積を

はめる

solid torus を得る

これで位相不変量が定まる．これを内積とすると

はめる

依存

によって T^* の元が定まり，

そこから とする事ができる．

Th.（Ocneanu 1991）

Tのcenter と $H_{S^1 \times S^1}$ が自然に同型である．

Summary in hep-th

Consider a 2d CFT with fusion category symmetry $\mathbf{F}$ on $S^1 \times \mathbb{R}_{\text{time}}$.
Its Hilbert space $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = \bigoplus_{\sigma\text{: simples of } Z(\mathbf{F})} \mathcal{H}_\sigma.$$

We're interested in

$$\text{``}\frac{\dim \mathcal{H}_\sigma}{\dim \mathcal{H}}\text{''} := \lim_{\beta \to 0} \frac{\mathrm{tr}_{\mathcal{H}_\sigma} e^{-\beta H}}{\mathrm{tr}_{\mathcal{H}} e^{-\beta H}}.$$

Summary in AQFT

Consider a chiral or non-chiral conformal net $\mathcal{A}$ with trivial $\mathbf{DHR}(\mathcal{A})$, acted on by a fusion category $\mathbf{F}$ on S^1.

It should have a unique natural Hilbert space associated; denote it by $\mathcal{H}$.

Let $\mathcal{B} = \mathcal{A}^{\mathbf{F}}$. Then there should be natural representations $\mathbb{H}_\sigma$ of $\mathcal{B}$ on S^1 for each simple σ of $\mathbf{DHR}(\mathcal{B}) = Z(\mathbf{F})$.

Let $\mathcal{B} \subset \mathcal{A}$ is specified by a Q-system $\theta \in \mathbf{DHR}(\mathcal{B})$. Then I believe

$$\mathcal{H} = \mathbb{H}_\theta = \bigoplus_{\substack{\sigma\text{: simples of}\\ \mathbf{DHR}(\mathcal{B})}} \langle \sigma, \theta \rangle \mathbb{H}_\sigma.$$

We're interested in

$$\text{``}\frac{\dim \mathbb{H}_{\sigma}}{\dim \mathbb{H}_{1}}\text{''} := \lim_{\beta \to 0} \frac{\mathrm{tr}_{\mathbb{H}_{\sigma}}\, e^{-\beta H}}{\mathrm{tr}_{\mathbb{H}_{1}}\, e^{-\beta H}}.$$

We conjecture that it is universally given by d_{σ}.

Can we/you prove it?

(Note that the net $\mathcal{A}$ disappeared from the formulation;
only $\mathcal{B} = \mathcal{A}^{\mathbf{F}}$ matters.)

When $\mathcal{B}$ is a rational chiral conformal net,
this question was raised in [Longo, gr-qc/9605073]
and more or less answered in [Kawahigashi-Longo, math-ph/0405037].

The point is that

$$\mathrm{tr}_{\mathbb{H}_\sigma}\, q^H = \mathrm{tr}_{\mathbb{H}_\sigma}\, q^{L_0 - c/24}$$

are the irreducible characters of the VOA associated to $\mathcal{B}$, and therefore

$$\mathrm{tr}_{\mathbb{H}_\sigma}\, e^{-\beta H} = \sum_\rho S_{\sigma\rho}\, \mathrm{tr}_{\mathbb{H}_\rho}\, e^{-(4\pi^2/\beta)H}$$

where $S_{\sigma\rho}$ is the S matrix of $\mathbf{DHR}(\mathcal{B})$.

In the $\beta \to 0$ limit, the term $\rho = 1$ dominates in the RHS, and therefore

$$\lim_{\beta\to 0} \frac{\mathrm{tr}_{\mathbb{H}_\sigma}\, e^{-\beta H}}{\mathrm{tr}_{\mathbb{H}_1}\, e^{-\beta H}} = \frac{S_{\sigma 1}}{S_{11}} = d_\sigma.$$

Our physics arguments in [Lin-Okada-Seifnashri-YT, 2208.05495] for general non-chiral 2d CFT were basically the same.

Physics arguments on a 2d CFT with fusion category symmetry $\mathbf{F}$ imply

$$\mathrm{tr}_{\mathbb{H}_\sigma} e^{-\beta H} = \sum_\rho S_{\sigma\rho} \, \mathrm{tr}_{\mathbb{H}_\rho} e^{-(4\pi^2/\beta) H},$$

where $S_{\sigma\rho}$ is the S matrix of $Z(\mathbf{F})$.

(This should also follow from a proper bordism formulation of 2d CFT with $\mathbf{F}$ symmetry.)

In the $\beta \to 0$ limit, the term $\rho = 1$ dominates in the RHS, and therefore

$$\lim_{\beta\to 0} \frac{\mathrm{tr}_{\mathbb{H}_\sigma} e^{-\beta H}}{\mathrm{tr}_{\mathbb{H}_1} e^{-\beta H}} = \frac{S_{\sigma 1}}{S_{11}} = d_\sigma.$$

How hard would it be to make it into an actual theorem?

Defining $\mathbb{H}_\sigma$ for a non-chiral net $\mathcal{B}$ seems possible. (In the chiral case it was done by [Carpi-Conti-Hillier-Weiner, 1202.2543]).

Studying $\mathrm{tr}_{\mathbb{H}_\sigma} e^{-\beta H}$ for non-chiral $\mathcal{B}$ should be doable. (In the chiral case it was done by [Longo-Tanimoto, 1608.08903].)

However, It seems very hard to prove

$$\mathrm{tr}_{\mathbb{H}_\sigma} e^{-\beta H} = \sum_\tau S_{\sigma\tau} \, \mathrm{tr}_{\mathbb{H}_\tau} e^{-(4\pi^2/\beta)H},$$

as we can't pass to VOA.

One would need to pass to full-VOAs. Even assuming it can be done, the representation theory of full VOAs is not well developed, so it wouldn't help us much, either.

That said, even in the chiral case, **mapping to VOA and using the S transformation seems too heavy-handed.**

The definition of d_σ via the sector theory,
or the Jones index, is supposed to quantity
the "ratio of dimensions" of ∞-dimensional representation spaces
of operator algebras.

And the representation $\mathbb{H}_\sigma$ of $\mathcal{B}$ on S^1 should arise via
a composition of a localized endomorphism σ on top of $\mathbb{H}_1$.

Shouldn't it be, then, that the property

$$\lim_{\beta\to 0} \frac{\mathrm{tr}_{\mathbb{H}_\sigma} e^{-\beta H}}{\mathrm{tr}_{\mathbb{H}_1} e^{-\beta H}} = d_\sigma$$

should follow from the very definition of d_σ using sector theory?

I don't know if it's very easy or very difficult,
due to my lack of understanding of basics of operator algebras.

Such a proof, if exists, would not only apply to

2d non-chiral conformal theories

but also to

higher-dimensional cases

with which I started my talk.

I hope that somebody in the audience comes up with an answer,
with whom I can write a paper together,
dedicated to Yasu and this occasion!